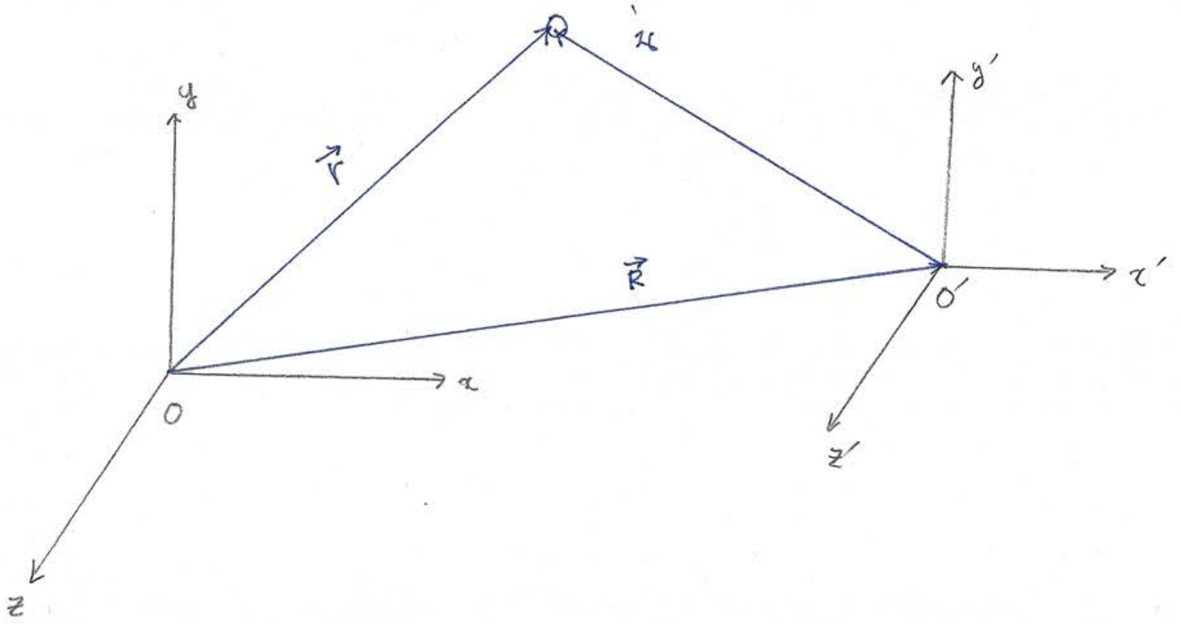
CH. Foundation of Special Relativity

§. Michelson - Morley Experiment (1887)

Conclusion: Speed of light is constant in all inertial frame.

§. Galilean transformation

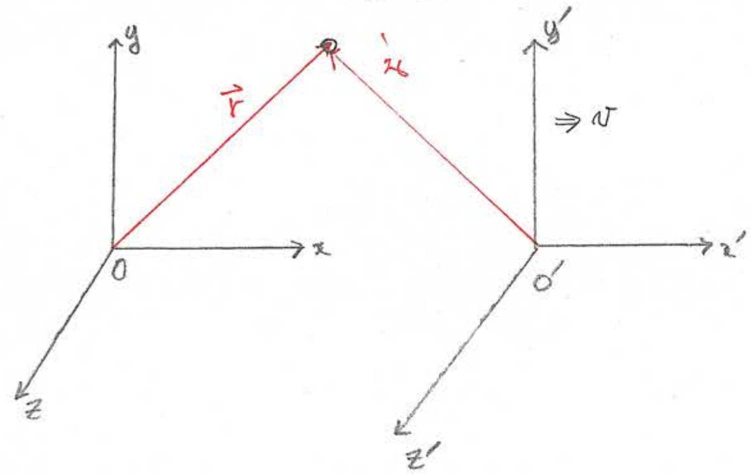


displacement vector at O $= \vec{r}$

displacement vector at O' $= \vec{r}'$

$$\vec{r} = \vec{R} + \vec{r}' \;;$$

Let $\vec{R} = vt\ \hat{x}$ $\left(\begin{cases} \hat{x} = \hat{x}' \\ \hat{y} = \hat{y}' \\ \hat{z} = \hat{z}' \end{cases} \quad v : \text{const} \right)$



$$\vec{r} = vt\,\hat{x} + \vec{r}'$$

$$\boxed{\begin{aligned} x &= x' + vt \\ y &= y' \\ z &= z' \\ t &= t' \end{aligned}}$$

Galilean Transf.

Can Galilean transformation explain result of Michelson - Morley Exp?

* O 에서 본 물체의 속력이 c (light velocity) 라 하자.

즉 $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = c^2$

Velocity from O' is

$$\left(\frac{dx'}{dt'}\right)^2 + \left(\frac{dy'}{dt'}\right)^2 + \left(\frac{dz'}{dt'}\right)^2$$

$$= \left(\frac{d}{dt}(x - vt)\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

$$= \left(\frac{dx}{dt} - v\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

$$= \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 + v^2 - 2v\frac{dx}{dt}$$

$$= c^2 + v\left(v - 2\frac{dx}{dt}\right)$$

$$\neq c^2 \qquad 7^{\forall}\ v$$

So Galilean transformation cannot explain

the result of Michelson - Morley Exp !!

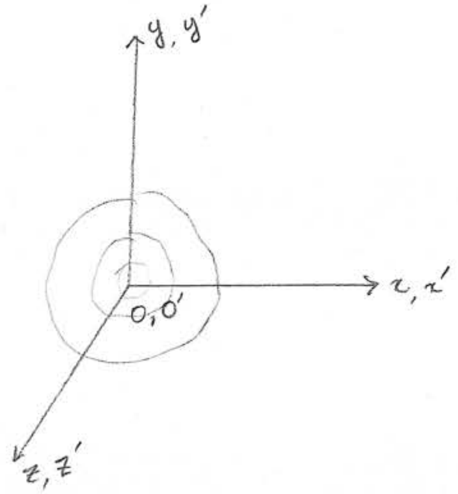
§ Einstein's Postulate for Special Relativity.

1. The laws of Physics are identical in all inertial frame

2. Speed of light is the same for all observers regardless of any relative motion of the source and observer

§. Lorentz Transformation

Consider a particular experiment in which a flash of light is sent out from O at the instant $t=0$ when the two origins O and O' coincide.



the front of light wave

$$x^2+y^2+z^2=c^2t^2 \quad \text{at } O \qquad (1)$$

From postulate (2)

$$x'^2+y'^2+z'^2=c^2t'^2 \quad \text{at } O' \qquad (2).$$

Put

$$\left.\begin{aligned} x' &= a_{11}x + a_{12}t \\ t' &= a_{21}x + a_{22}t \\ y' &= y \\ z' &= z \end{aligned}\right\} \qquad (3).$$

So Eq. (3) becomes

$$(a_{11}x + a_{12}t)^2 + y^2 + z^2 = c^2 (a_{21}x + a_{22}t)^2$$

$$\Rightarrow \left.\begin{aligned} &a_{11}a_{12} - c^2 a_{21}a_{22} = 0 \\ &a_{11}^2 - c^2 a_{21}^2 = 1 \\ &a_{12}^2 - c^2 a_{22}^2 = -c^2 \end{aligned}\right\} \quad (4).$$

Galilean Transf. $x' = x - vt$

Require

$$a_{11} ; a_{12} = 1 ; -v$$

$$\Rightarrow \frac{a_{12}}{a_{11}} = -v \quad (5).$$

Solving (4) and (5).

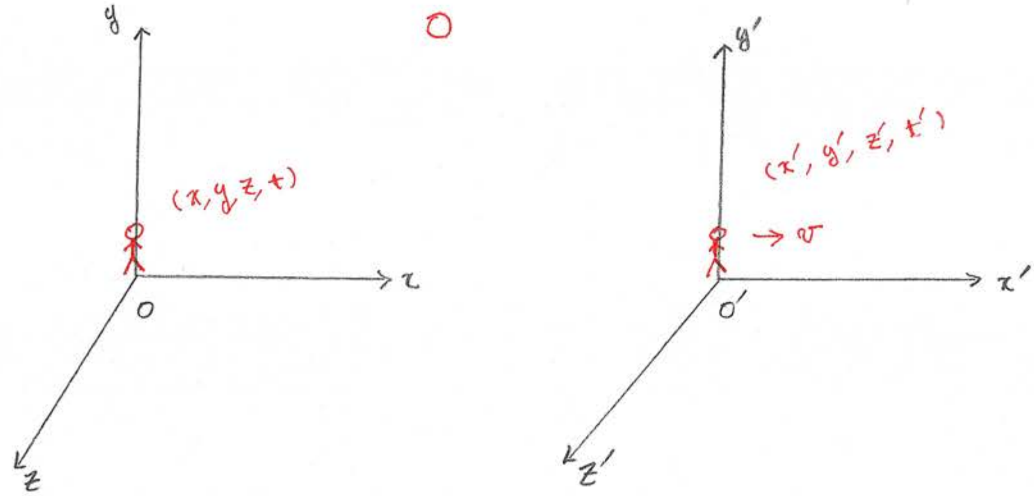
$$\left.\begin{aligned} &a_{11} = a_{22} = \gamma \\ &a_{12} = -\gamma v \\ &a_{21} = -\frac{\gamma v}{c^2} \end{aligned}\right\} \quad (6)$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (7).$$

$$\boxed{\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{vx}{c^2}\right) \end{aligned}}$$

Lorentz Transf.



Inversion

$$x = \gamma(x' + vt')$$

$$y = y'$$

$$z = z'$$

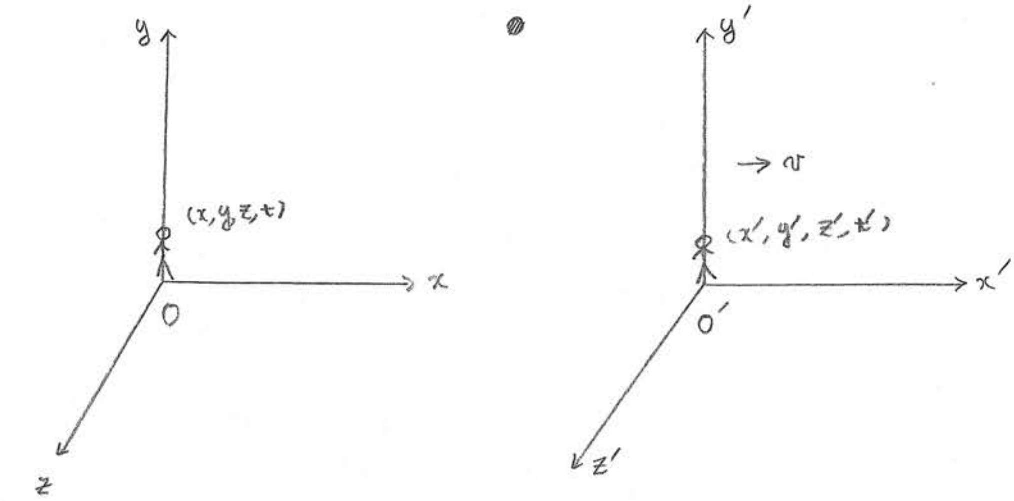
$$t = \gamma\left(t' + \frac{v}{c^2}x'\right)$$

note) $v \ll c$

$\gamma \approx 1$

$$\left.\begin{aligned} x' &\approx x - vt \\ y' &= y \\ z' &= z \\ t' &\approx t \end{aligned}\right) \quad \text{Galilean transf.}$$

§ Property of Lorentz transformation



$$\boxed{\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{v}{c^2}x\right) \end{aligned}}$$

Lorentz transformation

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

[1] $v << c$

$\gamma \approx 1$

$x' = x - vt$

$y' = y$

$z' = z$

$t' = t$ Galilean transformation

[2] Inverse Lorentz transformation

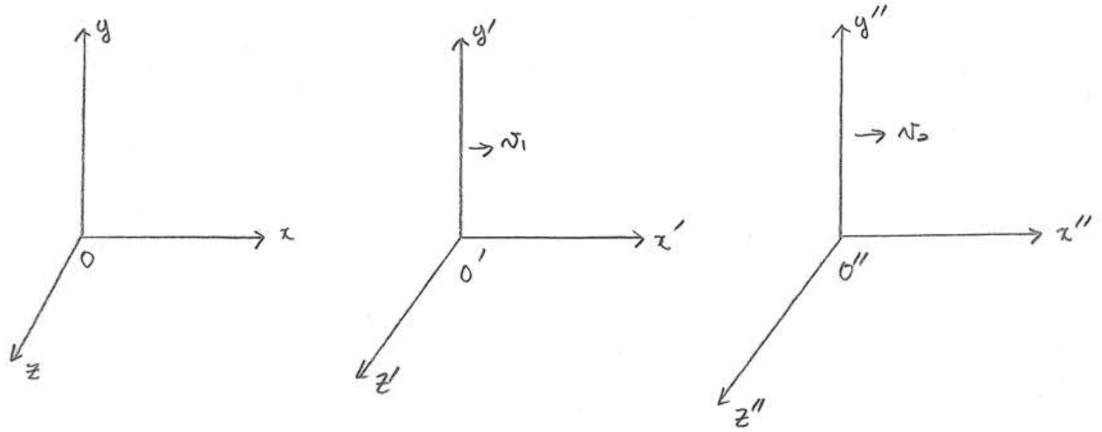
$x = \gamma (x' + vt')$

$y = y'$

$z = z'$

$t = \gamma (t' + \frac{v}{c^2} x')$

[3]



$$\begin{cases} x' = \gamma_1 (x - v_1 t) \\ y' = y \\ z' = z \\ t' = \gamma_1 \left(t - \dfrac{v_1}{c^2} x\right) \end{cases} \qquad \gamma_1 = \frac{1}{\sqrt{1 - \dfrac{v_1^2}{c^2}}}$$

$$\begin{cases} x'' = \gamma_2 (x' - v_2 t') \\ y'' = y' \\ z'' = z' \\ t'' = \gamma_2 \left(t' - \dfrac{v_2}{c^2} x'\right) \end{cases} \qquad \gamma_2 = \frac{1}{\sqrt{1 - \dfrac{v_2^2}{c^2}}}$$

$$\Rightarrow \begin{cases} x'' = \gamma (x - v t) \\ y'' = y \\ z'' = z \\ t'' = \gamma \left(t - \dfrac{v}{c^2} x\right) \end{cases} \qquad v = \frac{v_1 + v_2}{1 + \dfrac{v_1 v_2}{c^2}}$$

$$\gamma = \frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

* why ?

$$dx = \gamma\,(dx' + v\,dt')$$

$$dt = \gamma\,(dt' + \frac{v}{c^2}\,dx')$$

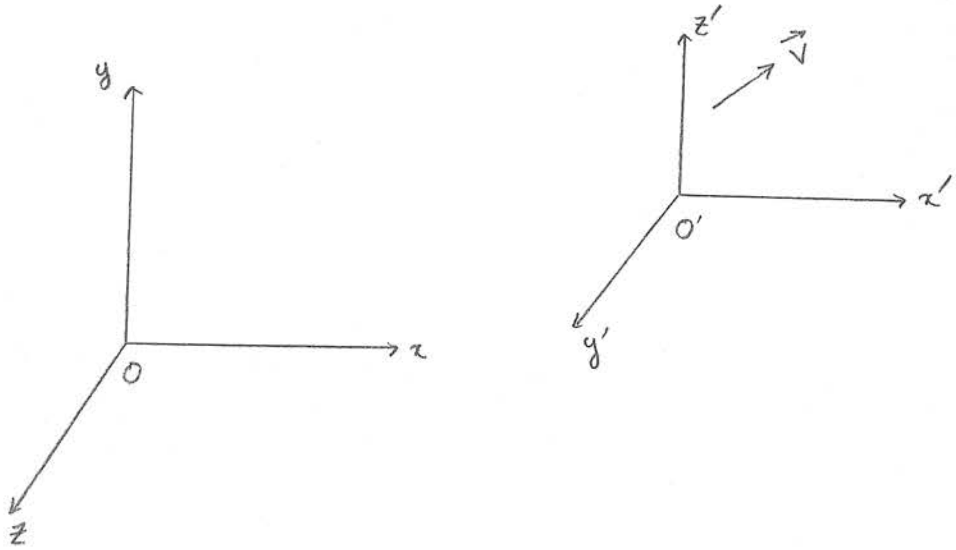
$$\Rightarrow \quad \frac{dx}{dt} = \frac{dx' + v\,dt'}{dt' + \frac{v}{c^2}\,dx'}$$

$$= \frac{\frac{dx'}{dt'} + v}{\frac{v}{c^2}\frac{dx'}{dt'} + 1}$$

O'' 가 O' 에 대하여 속도가 v_2 이니 O 가 O'' 를 볼때 속도는

$$\frac{dx}{dt} = \frac{v_2 + v_1}{\frac{v_1}{c^2}v_2 + 1} = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$$

[4] General Lorentz transformation



$$\vec{\beta} = \frac{\vec{V}}{c} \quad , \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\boxed{\begin{aligned} \vec{r}\,' &= \vec{r} + \frac{(\vec{\beta}\cdot\vec{r})\,\vec{\beta}}{\beta^2}(\gamma - 1) - \vec{\beta}\,\gamma c t \\ t' &= \gamma t - (\vec{\beta}\cdot\vec{r})\,\frac{\gamma}{c} \end{aligned}}$$

General Lorentz transf.

Ex1) $\vec{V} = (v, 0, 0)$

$$\vec{\beta}\cdot\vec{r} = \frac{v}{c}x$$

$$x' = x + \frac{\frac{\cancel{v}}{\cancel{c}}x}{\cancel{\beta^2}}\frac{\cancel{v}}{\cancel{c}}(\gamma-1) - \frac{v}{c}\gamma c t$$

$$= x + x(\gamma-1) - \frac{v}{\cancel{c}}\gamma\cancel{c}t$$

$$= \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma t - \frac{v}{c}x\frac{\gamma}{c}$$

$$= \gamma\left(t - \frac{v}{c^2}x\right)$$

$\Rightarrow$

$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

//

(ex2) $\vec{V} = (0, v, 0)$

$$\vec{\beta}\cdot\vec{r} = \frac{v}{c}y$$

$$x' = x$$

$$y' = y + \frac{\frac{v}{\cancel{c}}y}{\cancel{\beta^2}}\frac{v}{\cancel{c}}(\gamma-1) - \frac{v}{\cancel{c}}\gamma\cancel{c}t$$

$$= \gamma y - \gamma v t$$

$$= \gamma(y - vt)$$

$$z' = z$$

$$t' = \gamma t - \frac{v}{c}y\frac{\gamma}{c} = \gamma\left(t - \frac{v}{c^2}y\right)$$

$$\Rightarrow \quad x' = x$$

$$y' = \gamma(y - vt)$$

$$z' = z$$

$$t' = \gamma\left(t - \frac{v}{c^2}y\right)$$

*

Ex3) $\vec{V} = (v_1, v_2, 0)$

$$v^2 = v_1^2 + v_2^2$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \equiv \frac{1}{\sqrt{1-\frac{v_1^2+v_2^2}{c^2}}}$$

$$\vec{\beta}\cdot\vec{r} = \frac{v_1}{c}x + \frac{v_2}{c}y \ , \quad \beta^2 = \frac{v_1^2+v_2^2}{c^2}$$

$$\begin{cases} x' = \dfrac{\gamma v_1^2 + v_2^2}{v_1^2+v_2^2}\,x + (\gamma-1)\dfrac{v_1 v_2}{v_1^2+v_2^2}\,y - \gamma v_1 t \\[2ex] y' = (\gamma-1)\dfrac{v_1 v_2}{v_1^2+v_2^2}\,x + \dfrac{v_1^2+\gamma v_2^2}{v_1^2+v_2^2}\,y - \gamma v_2 t \\[2ex] z' = z \\[1ex] t' = \gamma\left(t - \dfrac{v_1}{c^2}x - \dfrac{v_2}{c^2}y\right) \end{cases}$$

* non-relativistic limit $(\gamma \to 1)$

$$x' = x - v_1 t$$

$$y' = y - v_2 t$$

$$z' = z$$

$$t' = t$$

Galilean transf. ※

[5] Lorentz invariance

A scalar quantity which is not changed by Lorentz transformation is called " Lorentz invariant quantity".

Consider two points (x_1, y_1, z_1, t_1) (x_2, y_2, z_2, t_2) at O system.

Then

$$\begin{cases} x_1' = \gamma(x_1 - vt_1) \\ y_1' = y_1 \\ z_1' = z_1 \\ t_1' = \gamma(t_1 - \frac{v}{c^2}x_1) \end{cases} \qquad \begin{cases} x_2' = \gamma(x_2 - vt_2) \\ y_2' = y_2 \\ z_2' = z_2 \\ t_2' = \gamma(t_2 - \frac{v}{c^2}x_2) \end{cases}$$

Then

$$(ct_2' - ct_1')^2 - (x_2' - x_1')^2 - (y_2' - y_1')^2 - (z_2' - z_1')^2$$

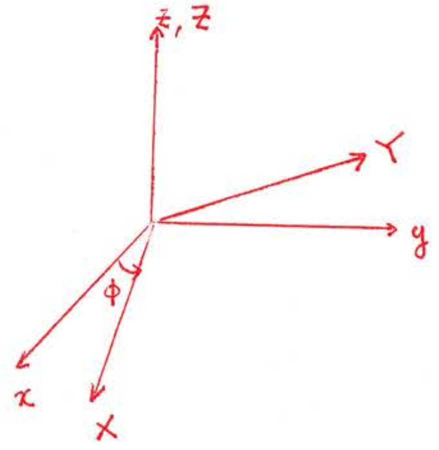
$$= (ct_2 - ct_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2$$

So

① $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ is Lorentz invariant

② $c^2t^2 - x^2 - y^2 - z^2$ is Lorentz invariant

* Coordinate Rotation



$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$X^2 + Y^2 + Z^2 = x^2 + y^2 + z^2$$

So, $x^2 + y^2 + z^2$ is rotation - invariant !!

Lorentz transformation is a kind of rotation ?

Yes !! rotation at 4-dimensional space - time

[6] 4-vector notation

Put $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = ict$

∘ $x' = \gamma(x - vt)$

$\Rightarrow x' = \gamma(x - \frac{v}{ic}\, ict)$

$\Rightarrow x_1' = \gamma x_1 + i\beta\gamma x_4$ $\qquad \beta = \frac{v}{c}$

∘ $t' = \gamma(t - \frac{v}{c^2} x)$

$\Rightarrow ict' = \gamma\, ict - i\gamma \frac{v}{c} x$

$\Rightarrow x_4' = \gamma x_4 - i\beta\gamma x.$

So Lorentz transformation is

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Let $\cos\phi = \gamma = \frac{1}{\sqrt{1-\beta^2}}$

$\sin^2\phi = 1 - \cos^2\phi = 1 - \frac{1}{1-\beta^2} = \frac{-\beta^2}{1-\beta^2}$

$\Rightarrow \sin\phi = i\beta\gamma$

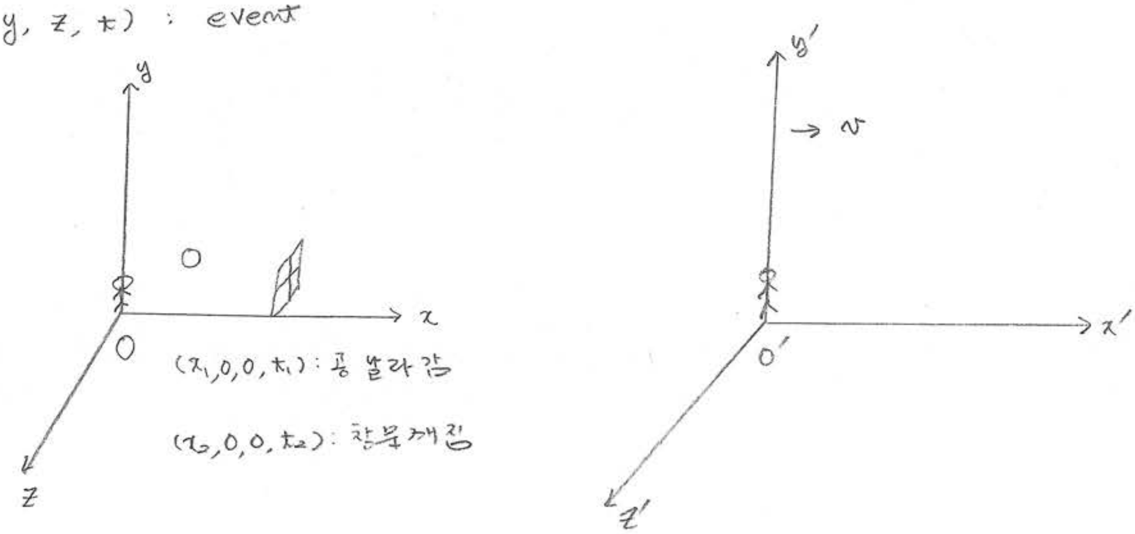
So

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} \cos\phi & 0 & 0 & \sin\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\phi & 0 & 0 & \cos\phi \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

So Lorentz transformation is rotation at 4-dimensional space-time.

[7] Causality

(x, y, z, t) : event



O ; U 공의속도 $\quad \Delta t = t_2 - t_1 > 0$

$$U \Delta t = x_2 - x_1$$

O' :

$$t_2' = \gamma \left(t_2 - \frac{v x_2}{c^2} \right)$$

$$t_1' = \gamma \left(t_1 - \frac{v x_1}{c^2} \right)$$

$$\Delta t' \equiv t_2' - t_1'$$

$$= \gamma (t_2 - t_1) \left[1 - \frac{v}{c^2} U \right]$$

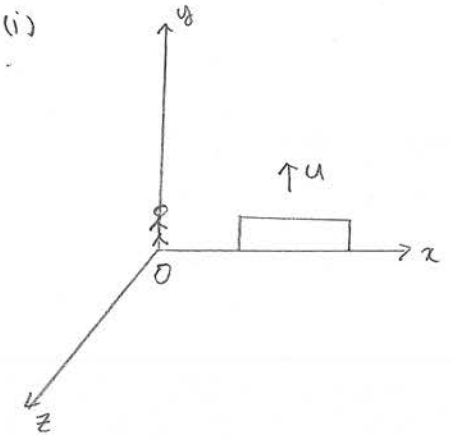
In order to be $\Delta t' \geq 0$

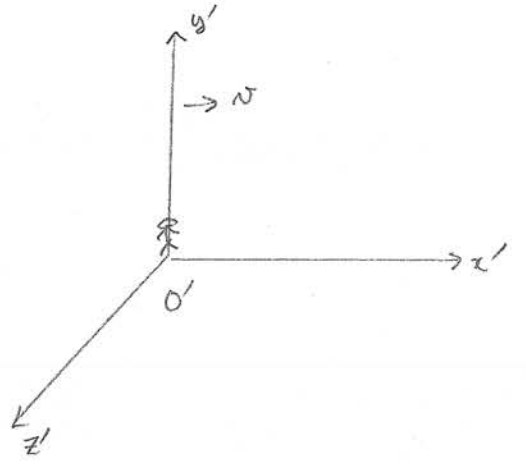
$$vU \leq c^2$$

So $\boxed{\text{every velocity} \leq c}$ Causality Condition

[8] Shape change Problem

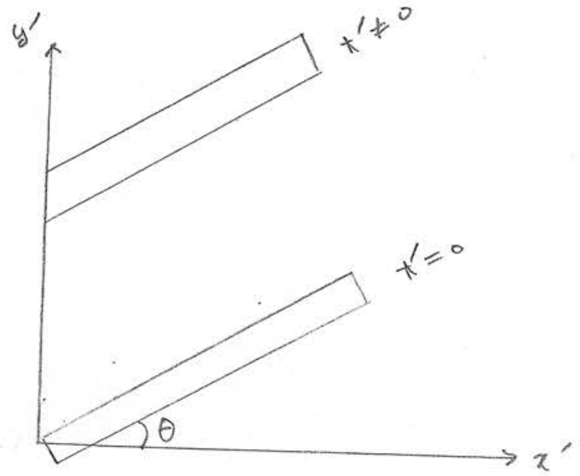
(i)





$O: \quad y = ut$

$O': \quad y' = y = ut = u\gamma\left(t' + \frac{v}{c^2}x'\right)$



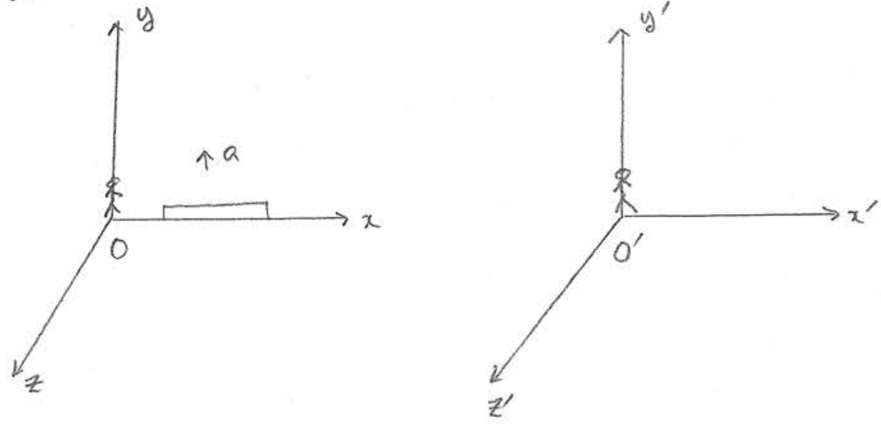
$$\tan\theta = \frac{u\gamma v}{c^2}$$

$$\theta = \tan^{-1}\frac{u\gamma v}{c^2}$$

Why? Synchronism (동시성)

질문 : O 에서 위로 운동 하는 물질이 O' 에서 보면 어떤 운동을 할까?

(ii)

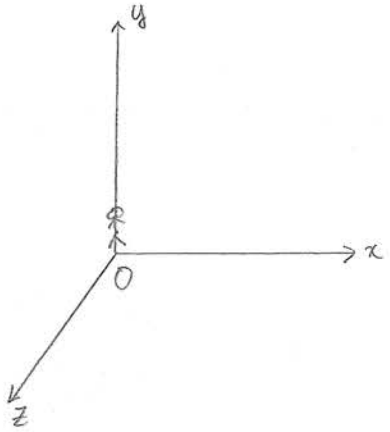


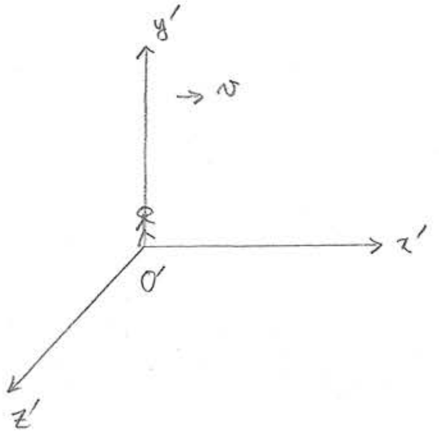
$$O: \quad y = \frac{1}{2} a t^2$$

$$O': \quad y' = y = \frac{1}{2} a t^2 = \frac{1}{2} a \gamma^2 \left(t' + \frac{v}{c^2} x' \right)^2$$

"Shape of rod becomes parabola"

[9] Minkowski diagram





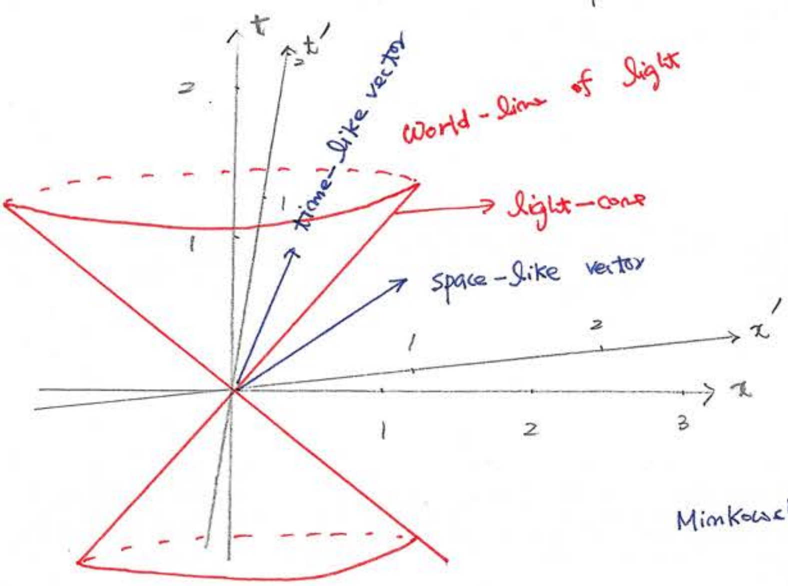
$$x' = \gamma(x - vt) \qquad x = \gamma(x' + vt')$$

$$y' = y \qquad y = y'$$

$$z' = z \qquad z = z'$$

$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \qquad t = \gamma\left(t' + \frac{v}{c^2}x'\right)$$

space - time



Minkowski diagram

(i) $x = 0,\ t = 0 \Rightarrow x' = 0,\ t' = 0$ Common origin

(ii) The past history, the present, and the future of a moving particle can be represented by a single curve in space-time.

This line is called "world-line" of the particle.

(iii) x - axis : $t=0$

t - axis : $x=0$

By same way

x'- axis : $t'=0$

$\Rightarrow$

$$t - \frac{v}{c^2}x = 0 \qquad t = \frac{v}{c^2}x$$

t'-axis : $x'=0$

$$x - vt = 0 \qquad t = \frac{1}{v}x$$

(iv) Scale:

$(t'=0,\ x'=1)$

$$x = \gamma > 1$$

$$t = \gamma \frac{v}{c^2}$$

$(t'=1,\ x'=0)$

$$t = \gamma > 1$$

$$x = \gamma v$$

(v) World line of light

$x = ct,\ x' = ct'$

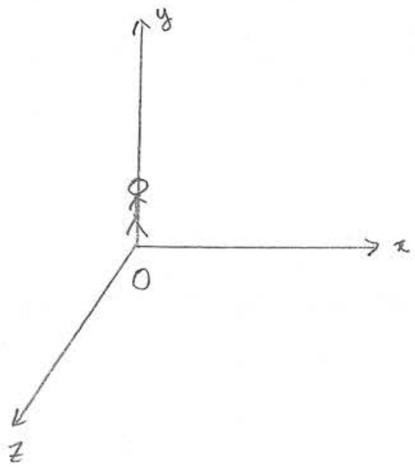
light - cone

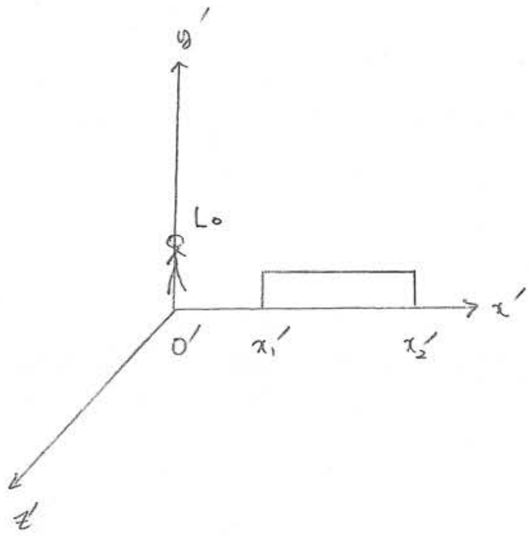
space - like : violate causality

time - like : preserve causality

CH. 2 Relativistic Kinematics

§ Length - contraction





O' : $L_0 = x_2' - x_1'$

O : $x_2' = \gamma(x_2 - v t_2)$

$x_1' = \gamma(x_1 - v t_1)$

$x_2' - x_1' = \gamma(x_2 - x_1) - \gamma v (t_2 - t_1)$

Assuming that observer O measures the positions of the ends of the rod at the same time. $(t_2 = t_1)$

So

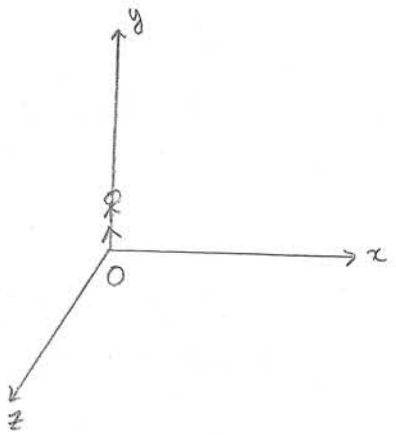
$L_0 = \gamma(x_2 - x_1)$

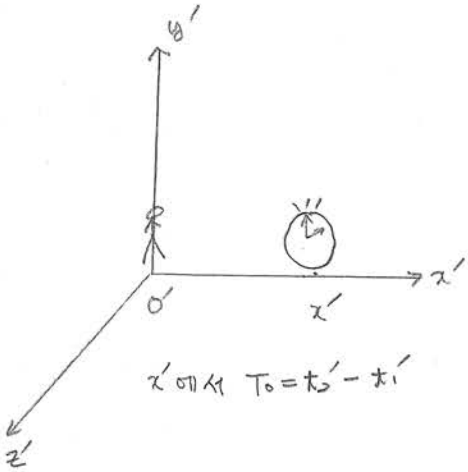
$$L = \frac{L_0}{\gamma} = \sqrt{1 - \frac{v^2}{c^2}}\, L_0 < L_0$$

Length - contraction !!

L_0 : rest - length

§ time - dilation





O': x' 에서 $T_0 = t_2' - t_1'$ interval

O: $t_2 = \gamma(t_2' + \frac{v}{c^2} x_2')$

$t_1 = \gamma(t_1' + \frac{v}{c^2} x_1')$

$t_2 - t_1 = \gamma(t_2' - t_1') + \frac{\gamma v}{c^2}(x_2' - x_1')$

Since $x_2' = x_1' = x'$,

$T \equiv t_2 - t_1) = \gamma T_0 \quad > T_0$

time - dilation

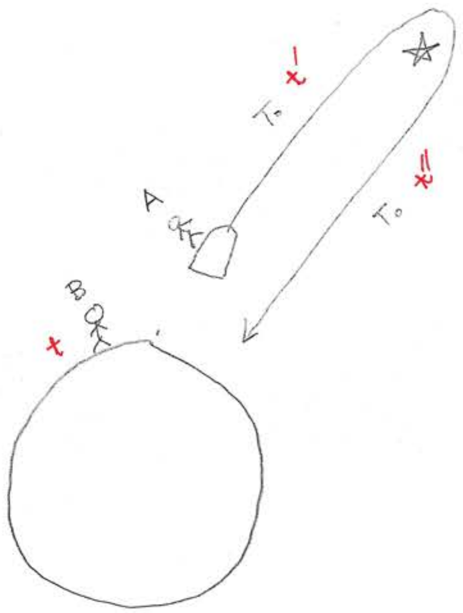
* Minkowski diagram 을 이용하여 length-contraction 과 time-dilation 을 설명하라.

§ Twin Paradox

Suppose a space traveler takes off on a journey to a distant star at a very high speed.

Let him travel for a certain length of time, say T_0, as measured by his own clock that he takes with him.

So total travel time for the trip is $2T_0$



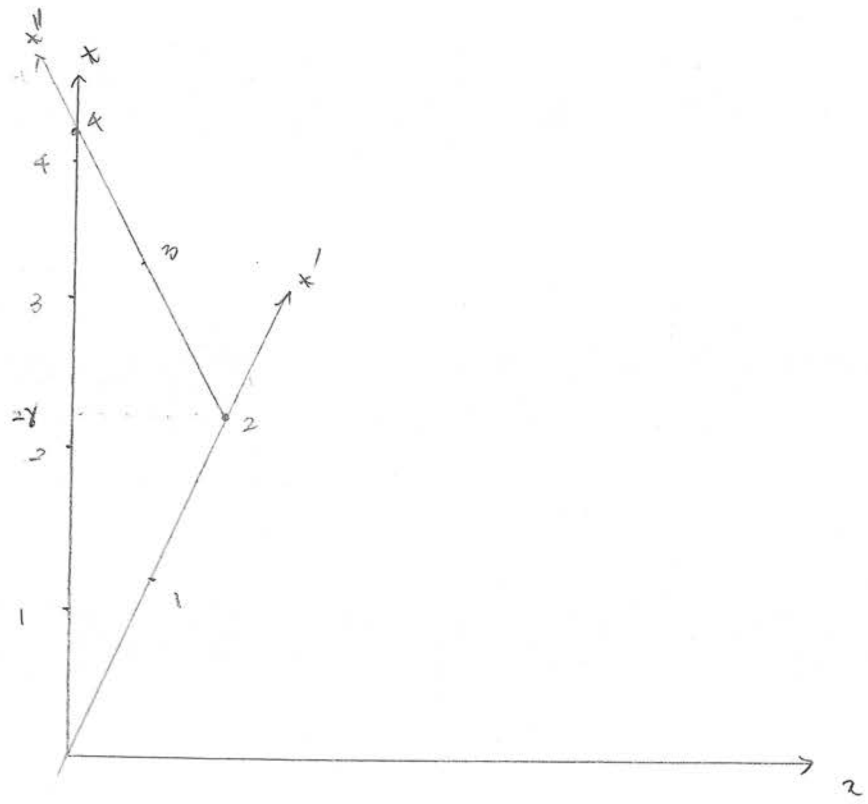
total travel time

A의 입장 { 지구 $\gamma\, 2T_0$ / 로케트 $2T_0$ } A가 더 젊다

B의 입장 { 지구 $\frac{2T_0}{\gamma}$ / 로케트 $2T_0$ } B가 더 젊다.

real journey : need "change of direction of velocity" and thereby undergoes a change from one reference frame to another.

Minkowski diagram

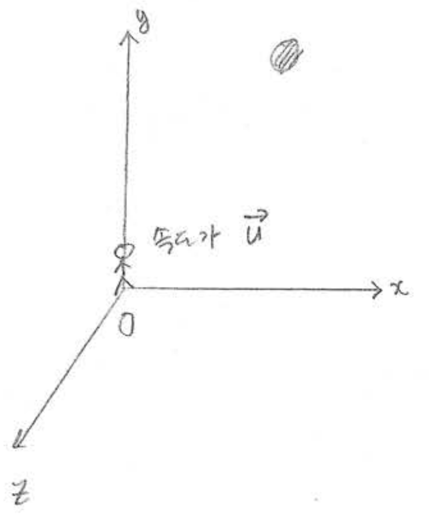


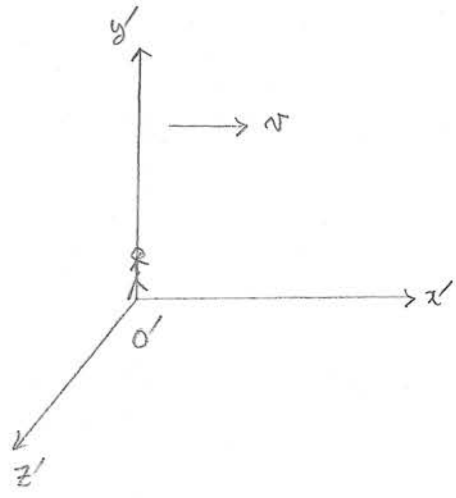
t' axis : $t = \frac{1}{v} x$

t'' axis : $t = -\frac{1}{v} x.$

A가 더 젊다 !!

§. velocity transformation





$$x' = \gamma(x - vt) \qquad x = \gamma(x' + vt')$$

$$y' = y \qquad y = y'$$

$$z' = z \qquad z = z'$$

$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \qquad t = \gamma\left(t' + \frac{v}{c^2}x'\right)$$

O: 물체의 속도가 $\vec{u} = (u_1, u_2, u_3)$

O' 계에서 본 속도는?

$$\frac{dx}{dt} = u_1 , \quad \frac{dy}{dt} = u_2 , \quad \frac{dz}{dt} = u_3$$

$$dx' = \gamma(dx - v\,dt)$$

$$dy' = dy$$

$$dz' = dz$$

$$dt' = \gamma\left(dt - \frac{v}{c^2}dx\right)$$

$$u_1' \equiv \frac{dx'}{dt'}$$

$$= \frac{dx - v\,dt}{dt - \frac{v}{c^2}dx}$$

$$= \frac{u_1 - v}{1 - \frac{v}{c^2}u_1}$$

$$u_2' \equiv \frac{dy'}{dt'}$$

$$= \frac{dy}{\gamma\left(dt - \frac{v}{c^2}dx\right)}$$

$$= \frac{u_2}{\gamma\left(1 - \frac{vu_1}{c^2}\right)}$$

$$u_3' = \frac{u_3}{\gamma\left(1 - \frac{vu_1}{c^2}\right)}$$

So

$$\boxed{\begin{aligned} u_1' &= \frac{u_1 - v}{1 - \frac{v u_1}{c^2}} \\ u_2' &= \frac{u_2}{\gamma\left(1 - \frac{v u_1}{c^2}\right)} \\ u_3' &= \frac{u_3}{\gamma\left(1 - \frac{v u_1}{c^2}\right)} \end{aligned}}$$

By same way

$$\boxed{\begin{aligned} u_1 &= \frac{u_1' + v}{1 + \frac{v u_1'}{c^2}} \\ u_2 &= \frac{u_2'}{\gamma\left(1 + \frac{v u_1'}{c^2}\right)} \\ u_3 &= \frac{u_3'}{\gamma\left(1 + \frac{v u_1'}{c^2}\right)} \end{aligned}}$$

Put

$$u = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

$$u' = \sqrt{u_1'^2 + u_2'^2 + u_3'^2}$$

Then

$$c^2 - u'^2 = \frac{c^2 (c^2 - u^2)(c^2 - v^2)}{(c^2 - \vec{u}\cdot\vec{v})^2}$$

our case

$$\frac{c^2 (c^2 - u^2)(c^2 - v^2)}{(c^2 - v u_1)^2}$$

If $u < c$, $v < c$, then $u' < c$

preserve causality !!

Also we can get the following useful relation

$$\boxed{\begin{aligned} \frac{\gamma(u')}{\gamma(u)} &= \gamma(v)\left[1 - \frac{u_1 v}{c^2}\right] \\ \frac{\gamma(u)}{\gamma(u')} &= \gamma(v)\left[1 + \frac{u_1' v}{c^2}\right] \end{aligned}}$$

Ex) Let $v = \frac{c}{2}$ $u_1' = \frac{c}{2}$, $u_2' = 0$, $u_3' = 0$

$$u_1 = \frac{c}{1 + \frac{1}{c^2}\frac{c}{2}\frac{c}{2}} = \frac{4}{5}c \quad , \quad u_2 = u_3 = 0$$

Ex) $u_1' = c$, $u_2' = u_3' = 0$

$$u_1 = \frac{v + c}{1 + \frac{1}{c^2} v \cdot c} = c \quad , \quad u_2 = u_3 = 0$$

Ex) Let's consider a particle moving in the (x, y) plane.

Let the velocity vector of the particle be inclined to the x-axis by an angle θ.

So $$\tan\theta = \frac{\dot{y}}{\dot{x}} = \frac{u_2}{u_1}$$

What is the angle at O' ?

$$\tan\theta' = \frac{u_2'}{u_1'} = \frac{\dfrac{u_2}{\gamma\left(1 - \frac{v u_1}{c^2}\right)}}{\dfrac{u_1 - v}{1 - \frac{v u_1}{c^2}}}$$

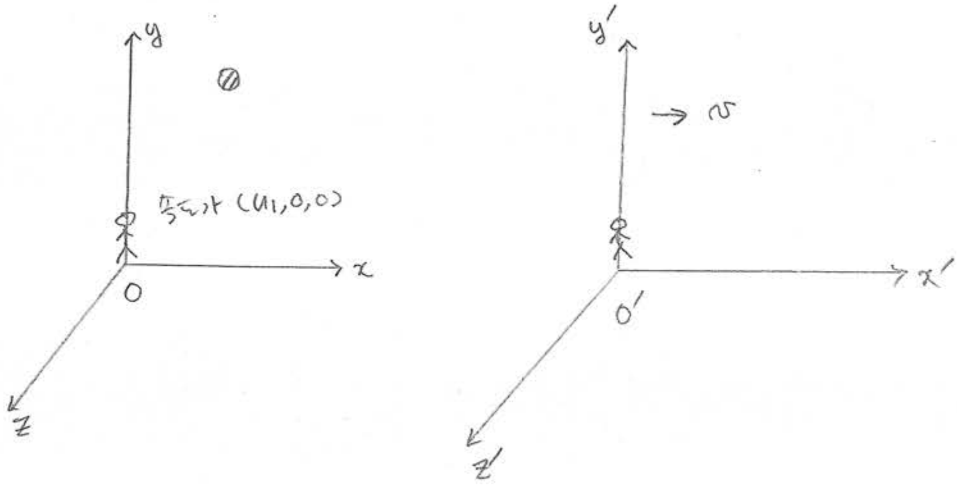
$$= \frac{u_2}{\gamma(u_1 - v)}$$

$$= \frac{\tan\theta}{\gamma(1 - v/u_1)}$$

note) In non-relativistic limit $\tan\theta' = \dfrac{\tan\theta}{1 - v/u_1}$

θ' is smaller relativistically than it would be non-relativistically.

8. Transformation of Linear acceleration



$x' = \gamma(x - vt)$

$y' = y$

$z' = z$

$t' = \gamma(t - \frac{v}{c^2}x)$

$x = \gamma(x' + vt')$

$y = y'$

$z = z'$

$t = \gamma(t' + \frac{v}{c^2}x')$

O: 물체의 속도 $\vec{U} = (u_1(t), 0, 0)$

$$\frac{du_1}{dt} \neq 0.$$

O'계에서 본 가속도는? $\frac{du_1'}{dt'} = ?$

Then

$$u_1' = \frac{u_1 - v}{1 - \frac{vu_1}{c^2}}$$

$$\frac{du_1'}{dt'} = \frac{d}{dt'}\left(\frac{u_1 - v}{1 - \frac{vu_1}{c^2}} \right)$$

$$= \frac{d}{dt}\left(\frac{u_1 - v}{1 - \frac{vu_1}{c^2}} \right) \frac{dt}{dt'}$$

$$= \frac{1}{\left(1 - \frac{vu_1}{c^2}\right)^2} \left[\frac{du_1}{dt}\left(1 - \frac{vu_1}{c^2}\right) - (u_1 - v)\left(-\frac{v}{c^2}\right)\frac{du_1}{dt} \right] \frac{dt}{dt'}$$

$$= \frac{1}{\left(1 - \frac{vu_1}{c^2}\right)^2} \left(1 - \cancel{\frac{vu_1}{c^2}} + \cancel{\frac{vu_1}{c^2}} - \frac{v^2}{c^2} \right) \frac{du_1}{dt} \frac{dt}{dt'}$$

$$= \frac{1}{\gamma^2 \left(1 - \frac{vu_1}{c^2}\right)^2} \frac{dt}{dt'} \frac{du_1}{dt} \qquad — ①$$

Since $dt' = \gamma\left(dt - \frac{v}{c^2} dx\right)$

$$\frac{dt'}{dt} = \gamma\left(1 - \frac{vu_1}{c^2}\right).$$

$$\therefore \frac{dt}{dt'} = \frac{1}{\gamma\left(1 - \frac{vu_1}{c^2}\right)} \qquad — ②$$

② → ①

$$\frac{du_1'}{dt'} = \frac{1}{\left[\gamma\left(1 - \frac{v u_1}{c^2}\right)\right]^3} \frac{du_1}{dt}$$

$$= \frac{1}{\left[\frac{\gamma(u')}{\gamma(u)}\right]^3} \frac{du_1}{dt} \Leftarrow \frac{\gamma(u')}{\gamma(u)} = \gamma(v)\left[1 - \frac{u_1 v}{c^2}\right]$$

$$\therefore \gamma^3(u') \frac{du_1'}{dt'} = \gamma^3(u) \frac{du_1}{dt}$$

$$\boxed{\left(1 - \frac{u_1'^2}{c^2}\right)^{-\frac{3}{2}} \frac{du_1'}{dt'} = \left(1 - \frac{u_1^2}{c^2}\right)^{-\frac{3}{2}} \frac{du_1}{dt}}$$

Lorentz invariance

Definition of Proper acceleration

$$\boxed{\alpha \equiv \left(1 - \frac{u^2}{c^2}\right)^{-\frac{3}{2}} \frac{du}{dt} = \frac{d}{dt}\left[\gamma(u)\, u\right]}$$

$$\frac{d}{dt}\left[\gamma(u)\right] = \frac{u}{c^2}\gamma^3(u)\frac{du}{dt}$$

if $\alpha = \text{const}$,

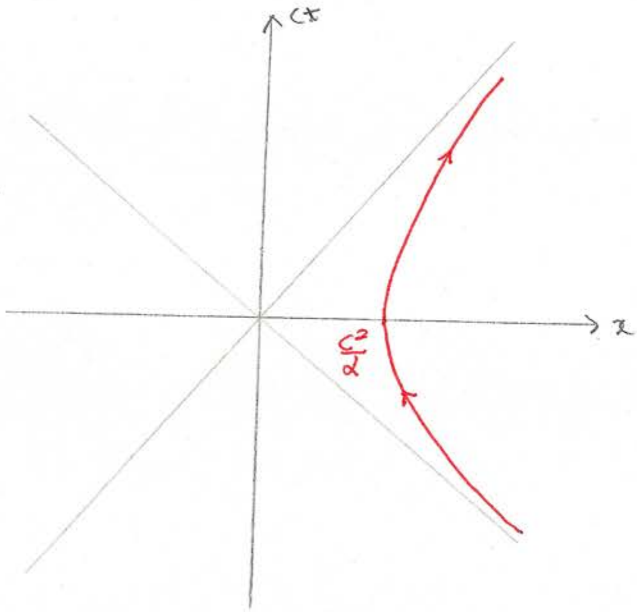
$$\gamma(u)\, u = \alpha t$$

$$\frac{u}{\sqrt{1 - \frac{u^2}{c^2}}} = \alpha t$$

$$u \equiv \frac{dx}{dt} = \frac{\alpha t}{\sqrt{1 + \frac{\alpha^2}{c^2} t^2}}$$

$$x = \frac{c^2}{\alpha}\sqrt{1 + \frac{\alpha^2}{c^2} t^2}$$

$$\Rightarrow \quad x^2 - c^2 t^2 = \frac{c^4}{\alpha^2}$$



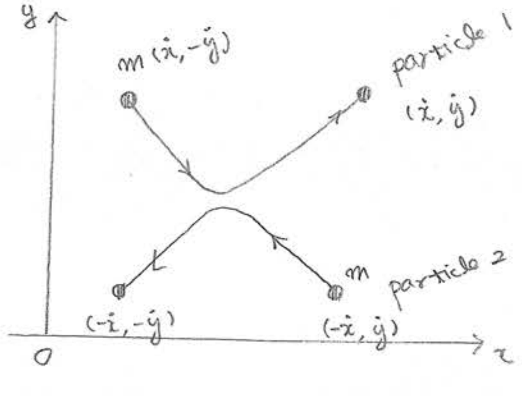
① $\alpha = \infty$, $x = \pm ct$ world line of light.

② $\frac{c^2}{\alpha}$ 뒤에서 쏜 빛은 particle을 따라잡을수 없다.

§. relativistic collision

Consider a collision of two identical particles.

Suppose the collision is perfectly elastic

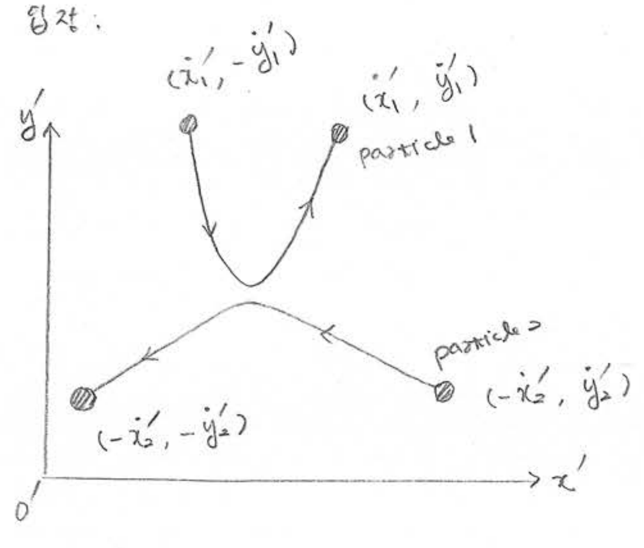


O 의 입장 :

	충돌 전의 속도	충돌 후의 속도
particle 1	$(\dot{x}, -\dot{y})$	$(\dot{x}, \dot{y})$
particle 2	$(-\dot{x}, \dot{y})$	$(-\dot{x}, -\dot{y})$

linear momentum is conserved !!

O′의 입장 :



	충돌전의 속도	충돌후의 속도
particle 1	$(\dot{x}_1', -\dot{y}_1')$	$(\dot{x}_1', \dot{y}_1')$
particle 2	$(-\dot{x}_2', \dot{y}_2')$	$(-\dot{x}_2', -\dot{y}_2')$.

Particle 1

$$\dot{x} = \frac{\dot{x}_1' + v}{1 + \frac{v\dot{x}_1'}{c^2}} \quad - ①$$

$$\dot{y} = \frac{\dot{y}_1'}{\gamma\left(1 + \frac{v\dot{x}_1'}{c^2}\right)} \quad - ②$$

particle 2

$$-\dot{x} = \frac{-\dot{x}_2' + v}{1 + \frac{v}{c^2}(-\dot{x}_2')} = \frac{-\dot{x}_2' + v}{1 - \frac{v}{c^2}\dot{x}_2'} \quad - ③$$

$$-\dot{y} = \frac{-\dot{y}_2'}{\gamma\left(1 + \frac{v}{c^2}(-\dot{x}_2')\right)} = \frac{-\dot{y}_2'}{\gamma\left(1 - \frac{v}{c^2}\dot{x}_2'\right)} \quad - ④$$

From ① & ③

$$\frac{\dot{x}_1' + v}{\dot{x}_2' - v} = \frac{1 + \frac{v\dot{x}_1'}{c^2}}{1 - \frac{v}{c^2}\dot{x}_2'} \quad - ⑤$$

From ② & ④

$$\frac{\dot{y}_1'}{\dot{y}_2'} = \frac{1 + \frac{v}{c^2}\dot{x}_1'}{1 - \frac{v}{c^2}\dot{x}_2'} \quad - ⑥$$

From 15 & 16

$$\frac{\dot{y}_1'}{\sqrt{1-\frac{v_1^2}{c^2}}} = \frac{\dot{y}_2'}{\sqrt{1-\frac{v_2^2}{c^2}}} \quad \text{— 17.}$$

$$v_1^2 = \dot{x}_1'^2 + \dot{y}_1'^2$$

$$v_2^2 = \dot{x}_2'^2 + \dot{y}_2'^2$$

(Pf)

y- component of linear momentum

충돌전

$$P_{i,y} = m\dot{y}_2' - m\dot{y}_1'$$

충돌후

$$P_{f,y} = m\dot{y}_1' - m\dot{y}_2'$$

$$\therefore P_{f,y} \neq P_{i,y}$$

To preserve the linear momentum conservation law, we re-define mass.

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma m_0$$

mass of moving particle

m_0 : rest mass

§. Mass - energy relation

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = m_0 \frac{d}{dt} \frac{\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}$$

Consider 1-dim case

$$dW = F\,dx = \frac{d}{dt}(m\dot{x})\,dx = \frac{dx}{dt}\,d(m\dot{x}) = \dot{x}\,d(m\dot{x})$$

$$W = \int dW$$

$$= \int \underbrace{\dot{x}}_{u}\ \underbrace{d(m\dot{x})}_{v'}$$

$$= m\dot{x}^2 - \int m\dot{x}\,d\dot{x}$$

$$= mv^2 - m_0 \int_0^v \frac{\dot{x}}{\sqrt{1-\dot{x}^2/c^2}}\,d\dot{x}$$

$$= mv^2 + m_0 c^2 \left[\sqrt{1-\dot{x}^2/c^2}\right]_0^v$$

$$= mv^2 + m_0 c^2 \left[\sqrt{1-\frac{v^2}{c^2}} - 1\right]$$

$$= mv^2 + m_0 c^2 \sqrt{1-\frac{v^2}{c^2}} - m_0 c^2$$

$$= mv^2 + mc^2\left(1-\frac{v^2}{c^2}\right) - m_0 c^2$$

$$= mc^2 - m_0 c^2 = (\gamma - 1)\, m_0 c^2$$

$\therefore$

$$T = mc^2 - m_0 c^2$$

$E = (mc^2), (\gamma \neq m_0 c^2)$

$\Rightarrow$ non-relativistic limit

$$T = \frac{1}{2} m_0 v^2 + \frac{3}{8} m_0 \frac{v^4}{c^2} + \cdots$$

§ inelastic scattering

classical mechanics

$$mv = 2mv'$$

$$v' = \frac{v}{2}$$

energy

$$E_i = \frac{1}{2} m v^2$$

$$E_f = \frac{1}{2}(2m)\left(\frac{v}{2}\right)^2 = \frac{m}{4} v^2$$

$$Q = E_i - E_f = \frac{m}{4} v^2$$; thermal energy

$$u_1' = \frac{u_1 - v}{1 - \frac{v u_1}{c^2}}$$

$$u_2' = \frac{u_2}{\gamma(v)\left[1 - \frac{v u_1}{c^2}\right]}$$

$$u_3' = \frac{u_3}{\gamma(v)\left[1 - \frac{v u_1}{c^2}\right]}$$

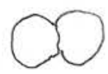
$$\frac{\gamma(u')}{\gamma(u)} = \gamma(v)\left[1 - \frac{u_1 v}{c^2}\right]$$

$$\frac{\gamma(u)}{\gamma(u')} = \gamma(v)\left[1 + \frac{u_1' v}{c^2}\right]$$

relativistic mechanics

O: 충돌전

충돌후



at rest

m_0 : rest mass

linear momentum is conserved!!

O': 충돌전

At rest

충돌후

rest $\overline{m}_0$

From Lorentz transformation

$$-v' = \frac{\dot{x} - v}{1 - \frac{v}{c^2}\dot{x}} = \frac{-2v}{1 + \frac{v^2}{c^2}}$$

$$* \quad \gamma(v') = \frac{1 + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}$$

linear momentum

충돌전 $\quad -m_0 \gamma(v') v'$

$$= -m_0 \frac{1}{\sqrt{1 - \frac{v'^2}{c^2}}} v'$$

충돌후 $\quad -\overline{m}_0 \gamma(v) v$

$$\overline{m}_0 = m_0 \frac{\gamma(v')}{\gamma(v)} \frac{v'}{v} \Rightarrow \frac{2}{1+\frac{v^2}{c^2}}$$

$$= m_0 \frac{1}{\gamma(v)} \frac{\cancel{1+\frac{v^2}{c^2}}}{1-\frac{v^2}{c^2}} \frac{2}{\cancel{1+\frac{v^2}{c^2}}}$$

$$= 2m_0 \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= 2m_0 \, \gamma(v)$$

In order to require the momentum conservation $\bar{m}_0 \neq 2m_0$

$$-\bar{m}_0 \gamma(v) v = -m_0 \gamma(v') v'$$

$$\Rightarrow \bar{m}_0 = 2m_0 \gamma(v)$$ ⇐ 계산해야 함.

$$= 2m_0 + \underline{2m_0[\gamma(v) - 1]} = 2m_0 + \Delta m$$

열로 변환된 ~~[illegible]~~ 질량

O: $E_i = 2m_0(\gamma(v) - 1)c^2$

$E_f = 0$

$E_i - E_f = \Delta m c^2$

O': 발표

$E_i = m_0[\gamma(v') - 1]c^2$

$E_f = \bar{m}_0[\gamma(v) - 1]c^2$

$E_i - E_f = \Delta m c^2$

§ 4-Vector notation

Lorentz transform

$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

put

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = ict$$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

where

$$\beta = \frac{v}{c}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = (1 - \beta^2)^{-\frac{1}{2}}$$

$$\Rightarrow \quad x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_1'^2 + x_2'^2 + x_3'^2 + x_4'^2$$

$$\frac{\gamma(u')}{\gamma(u)} = \gamma(v)\left[1 - \frac{u_1 v}{c^2}\right]$$

$$\gamma(u') = \gamma(u)\gamma(v)\left[1 - \frac{u_1 v}{c^2}\right]$$

① $V_1' = \gamma(u') u_1'$

$$= \gamma(u') \frac{u_1 - v}{1 - \frac{v u_1}{c^2}}$$

$$= \gamma(u)\gamma(v)(u_1 - v)$$

$$= \gamma(v)\underbrace{\left[\gamma(u) u_1\right]}_{V_1} - v\gamma(v)\frac{1}{ic}\underbrace{ic\gamma(u)}_{V_4}$$

$$= \gamma(v) V_1 + i\beta\gamma(v) V_4$$

② $V_2' = \gamma(u') u_2'$

$$= \gamma(u') \frac{u_2}{\gamma(v)\left[1 - \frac{v u_1}{c^2}\right]}$$

$$= \gamma(u) u_2 = V_2$$

③ $V_3' = V_3$

④ $V_4' = \gamma(u') ic$

$$= \gamma(v)\gamma(u) ic\left[1 - \frac{u_1 v}{c^2}\right]$$

$$= \gamma(v)\left[ic\gamma(u)\right] - i\frac{v}{c}\gamma(v)\left[\gamma(u) u_1\right]$$

$$= -i\beta\gamma(v) V_1 + \gamma(v) V_4$$

○ Four vector

Four vector A_μ $(\mu=1,2,3,4)$;

; A set of four quantities that transform in the same way as (x_1, x_2, x_3, x_4) under Lorentz transformation.

$$\begin{pmatrix} A_1' \\ A_2' \\ A_3' \\ A_4' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix}$$

So, $$\sum_{\mu=1}^{4} A_\mu'^2 = \sum_{\mu=1}^{4} A_\mu^2$$

○ Velocity 4 vector

$\left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}, \frac{dx_4}{dt}\right)$ is not 4-vector

⇒ Check Lorentz transform !!

Let

$$\vec{u} = \frac{dx_1}{dt}\hat{x}_1 + \frac{dx_2}{dt}\hat{x}_2 + \frac{dx_3}{dt}\hat{x}_3$$

Then

$$\boxed{\begin{aligned} V_1 &= \gamma(u)\,\dot{x}_1 \\ V_2 &= \gamma(u)\,\dot{x}_2 \\ V_3 &= \gamma(u)\,\dot{x}_3 \\ V_4 &= \gamma(u)\,ic \\ &\text{where} \\ \gamma(u) &= \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \end{aligned}}$$

Velocity 4-vector

숙제

$$\begin{pmatrix} V_1' \\ V_2' \\ V_3' \\ V_4' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix}$$

임을 증명하라 !!

Define proper time : τ

$$d\tau = \frac{dt}{\gamma(u)}$$

$$d\tau^2 = \frac{1}{c^2}\left[c^2dt^2 - dx^2 - dy^2 - dz^2\right]$$

Then

$$V_1 = \frac{dx_1}{d\tau}, \quad V_2 = \frac{dx_2}{d\tau}, \quad V_3 = \frac{dx_3}{d\tau}, \quad V_4 = \frac{dx_4}{d\tau}$$

$$\Rightarrow \boxed{V_\mu = \frac{dx_\mu}{d\tau}}$$ four-vector notation

$$\sum_{\mu=1}^{4} V_\mu^2 = -c^2$$

- momentum 4-vector

$$P_\mu = m_0 V_\mu$$

$$P_1 = m_0 V_1 = m_0 \gamma(u)\, \dot{x}_1$$

$$P_2 = m_0 V_2 = m_0 \gamma(u)\, \dot{x}_2$$

$$P_3 = m_0 V_3 = m_0 \gamma(u)\, \dot{x}_2$$

Since

$$m = m_0 \gamma(u),$$

$$P_1 = m \dot{x}_1$$

$$P_2 = m \dot{x}_2$$

$$P_3 = m \dot{x}_3$$

$$P_4 = m_0 V_4 = i c m_0 \gamma(u) = i c m$$

So

$$P_1^2 + P_2^2 + P_3^2 + P_4^2$$

$$= - m_0^2 c^2$$

$$= \vec{p}^{\,2} - m^2 c^2$$

$$\Rightarrow m^2 c^2 = \vec{p}^{\,2} + m_0^2 c^2$$

$$m^2 c^4 = \vec{p}^{\,2} c^2 + m_0^2 c^4.$$

Since

$$E = mc^2,$$

$$E^2 = \vec{p}^2 c^2 + m_0^2 c^4$$

Classical Quantum Mechanics

$$E = \frac{\vec{p}^2}{2m} + V$$

$$\Rightarrow i\hbar \frac{\partial}{\partial t} \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi$$ Schrödinger Eq.

Relativistic Quantum Mechanics

$$E^2 = \vec{p}^2 c^2 + m_0^2 c^4$$

$$-\hbar^2 \frac{\partial^2}{\partial t^2} \psi = -\hbar^2 c^2 \nabla^2 \psi + m_0^2 c^4 \psi$$ Klein - Gordon Eq.

$$\vec{p} = -i\hbar \vec{\nabla}$$

○ Force 4-vector

$$F_\mu \equiv \frac{dP_\mu}{d\tau}$$

$$P_\mu \equiv m_0 \frac{dx_\mu}{d\tau} \equiv m \frac{dx_\mu}{dt}$$

$$F_\mu = \gamma(u) \frac{d}{dt}\left(m \frac{dx_\mu}{dt}\right)$$

Let $\vec{f}$ be ordinary force.

$$F_1 = \gamma(u) f_1$$

$$F_2 = \gamma(u) f_2$$

$$F_3 = \gamma(u) f_3$$

$$F_4 = \gamma(u)\, ic \frac{dm}{dt}$$

From

$$\sum_\mu P_\mu P_\mu = -m_0^2 c^2$$

$$\Rightarrow \sum_\mu P_\mu F_\mu = 0$$

$$\Rightarrow P_1F_1 + P_2F_2 + P_3F_3 = -P_4F_4$$

$$\cancel{m}\,\cancel{\gamma(u)}\,\vec{v}\cdot\vec{f} = -ic\cancel{m}\, ic\, \cancel{\gamma(u)} \frac{dm}{dt}.$$

$$\vec{v}\cdot\vec{f} = c^2 \frac{dm}{dt} = \frac{d}{dt}(mc^2) = \frac{dE}{dt}$$

Power is variation of energy

CH2. 파동의 입자성

§ 전자기파 (Electromagnetic Wave)

1864년 Maxwell

빛 = 전자기파

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \sim 3 \times 10^8 \ (m/sec)$$

$$\frac{1}{4\pi\epsilon_0} \sim 9 \times 10^9 \ (Nm^2/C^2)$$

ϵ_0: 진공유전율 (vacuum permittivity)

$(8.85 \times 10^{-12} \ C^2/Nm^2)$

μ_0: 진공투자율 (vacuum permeability)

$(4\pi \times 10^{-7} \ Tm/A)$

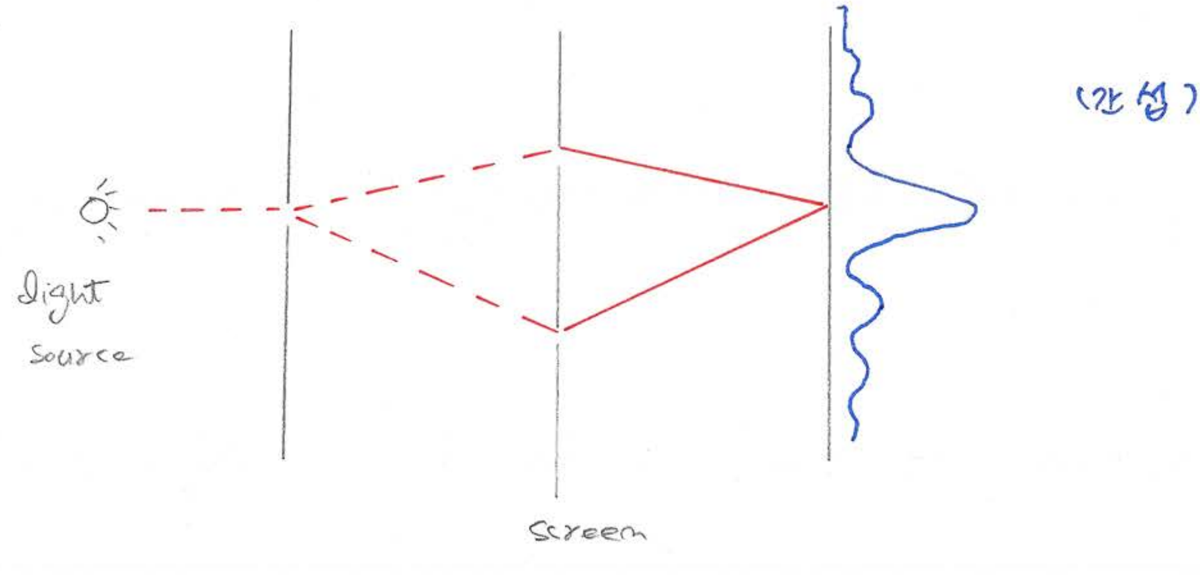
* 단위: $T = N/Am$, $A = \frac{C}{sec}$

(radio) 단파	(T.V) μ-wave	적외선	빨 보 가시광선	자외선	X-선	γ-선

→ ν

2

1801년 : Young의 실험



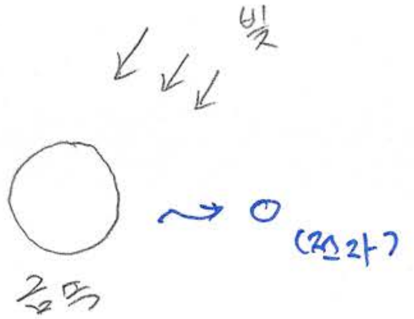
빛 = 파동 (1801년)

Young

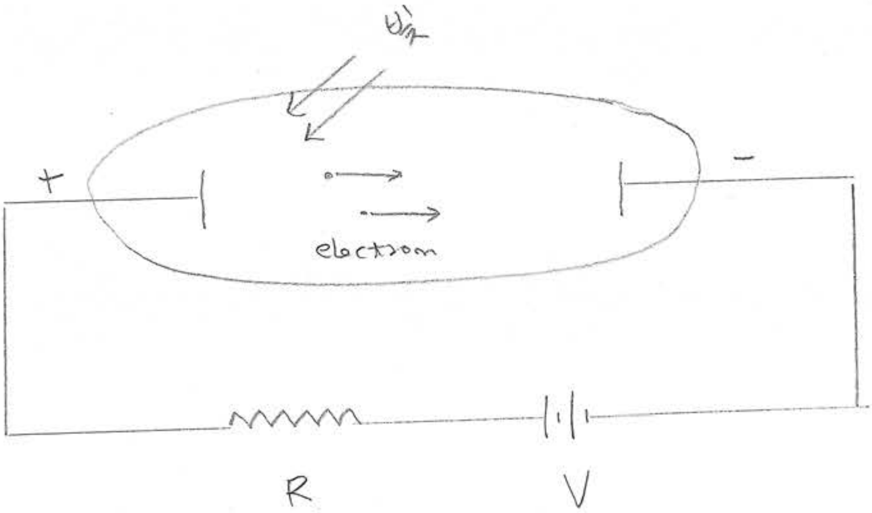
빛 = 전자기파 (1864년)

Maxwell

⊙ 광전효과 (photoelectric effect)



: 광전효과



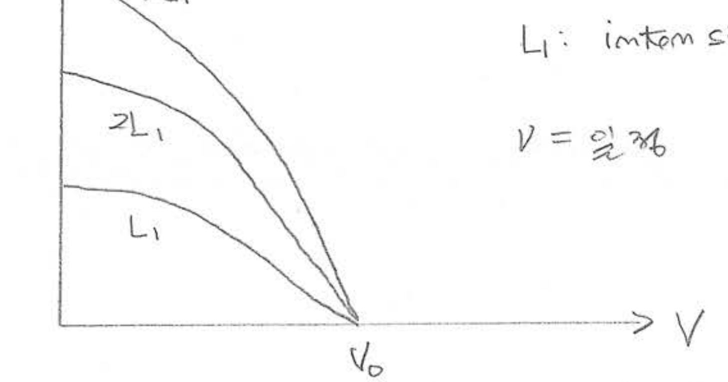
$V < V_0$: 광전전류 (photoelectron current)

$V > V_0$: no current

V_0: 소멸전압 (retarding potential)

stopping potential

$I \propto$ intensity



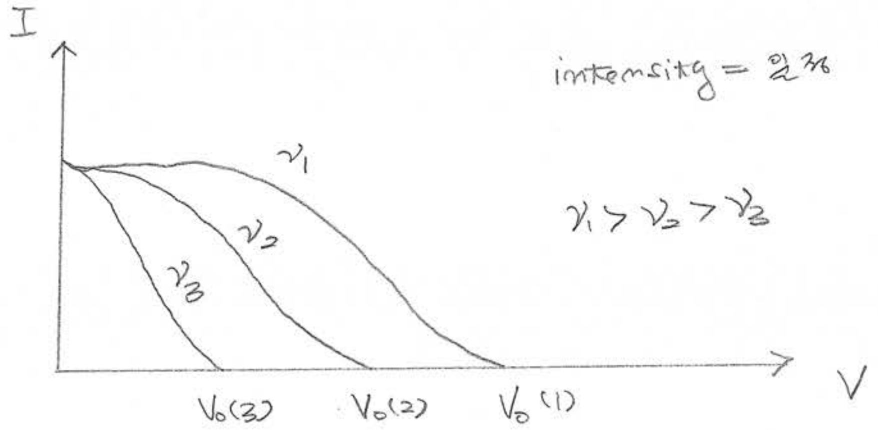
L_1: intensity

ν = 일정

(same retarding potential)

interpretation:

intensity $\propto$ # of photoelectron



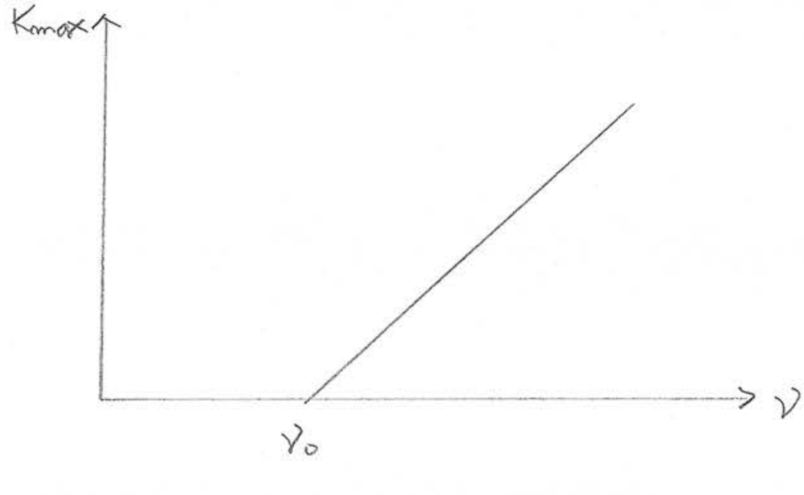
interpretation.

frequency $\propto$ energy of photoelectron

Einstein;

K_{max} : maximum energy of photoelectron

ν : frequency



$\therefore$

$$K_{max} = h(\nu - \nu_0)$$

$h = 6.626 \times 10^{-34}$ J·sec (Planck constant)

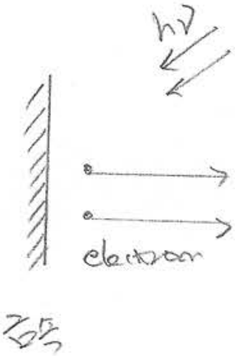
ν_0 : 문턱 진동수 (threshold frequency)

1905년 Einstein

빛 = photon (광자) $\Rightarrow$ 입자

$$E(\text{photon}) = h\nu$$

$$\Rightarrow \boxed{h\nu = h\nu_0 + K_{max}}$$



$$K_{max} = h\nu - h\nu_0$$

$h\nu_0$: work energy (or work function) (see 표 2.1)

photon energy	=	work energy	+	maximum electron energy

eV : electron volt

$$1\,\text{eV} = 1.6 \times 10^{-19}\ (\text{Joule})$$

$$h = 6.63 \times 10^{-34} \text{ joule sec}$$

$$= \frac{6.63 \times 10^{-34} \text{ joule sec}}{1.6 \times 10^{-19} \ \frac{\text{joule}}{\text{eV}}}$$

$$= 4.14 \times 10^{-15} \text{ eV} \cdot \text{sec}$$

$$\therefore \boxed{h = 6.63 \times 10^{-34} (\text{joule sec}) = 4.14 \times 10^{-15} \text{ eV} \cdot \text{sec}}$$

⇒ 예제 (p 66) : 생략

§ 빛의 이중성 (duality)

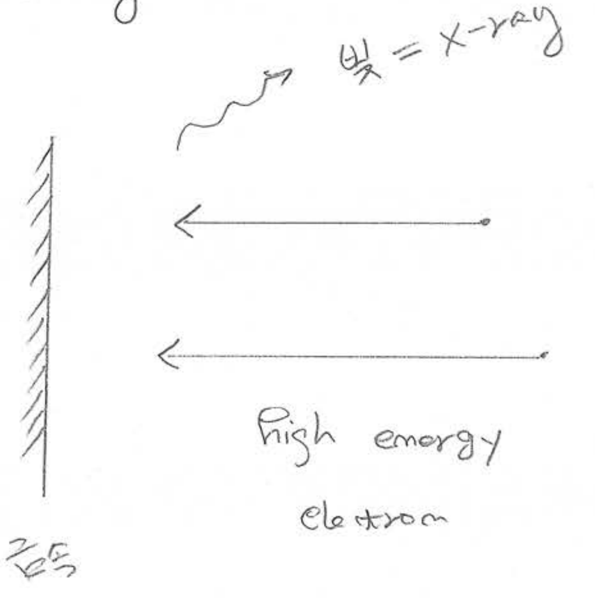
빛 = { Maxwell: electromagnetic wave (Ex) Yong 실험)
　　 { Einstein: photon (Ex: photoelectro effect)

⇒ 이 두가지 현상을 설명하기 위하여 "양자론" 등장

입자 = wave packet (ψ)

$|\psi|^2$: 입자의 존재 확률

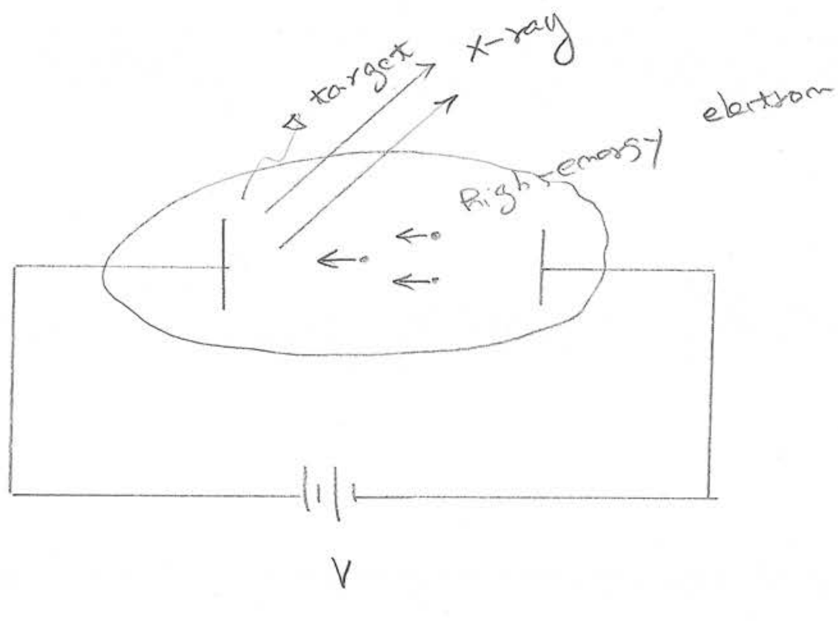
§ X-ray



1895년
Roemtgem
(뢴트겐)

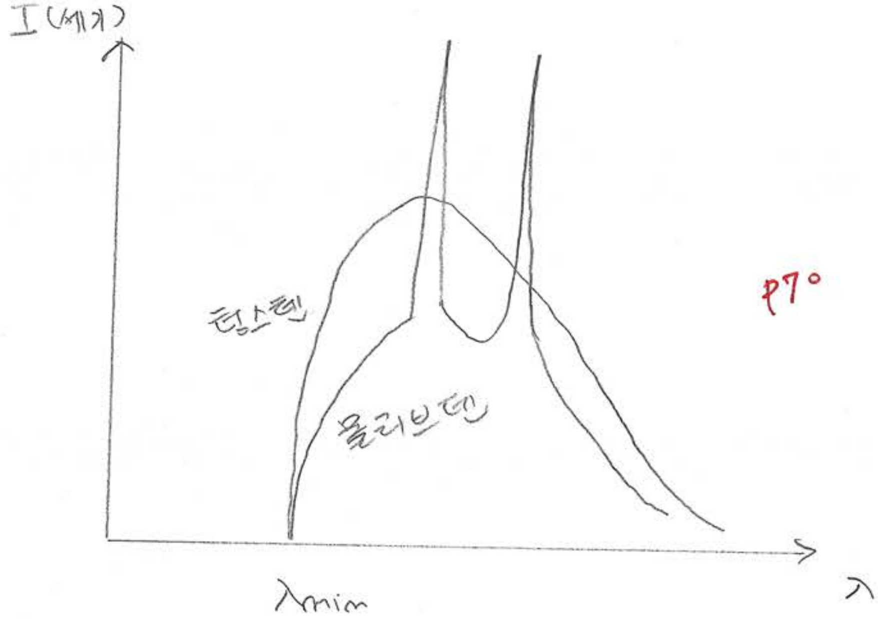
1912년 Max vom Laue: X-ray 파장 측정

X-ray 파장 ~ 0.013 ~ 0.048 nm



Ex) $V = 3.5\ kV$

target = { 텅스텐, 몰리브덴



P70

$$\lambda_{min} = \frac{1.24 \times 10^{-6}\ (V\ m)}{V}$$

이 식은 광전효과로 설명이 가능하다.

$$eV + h\nu_0 = h\nu_{max} = h\frac{c}{\lambda_{min}}$$

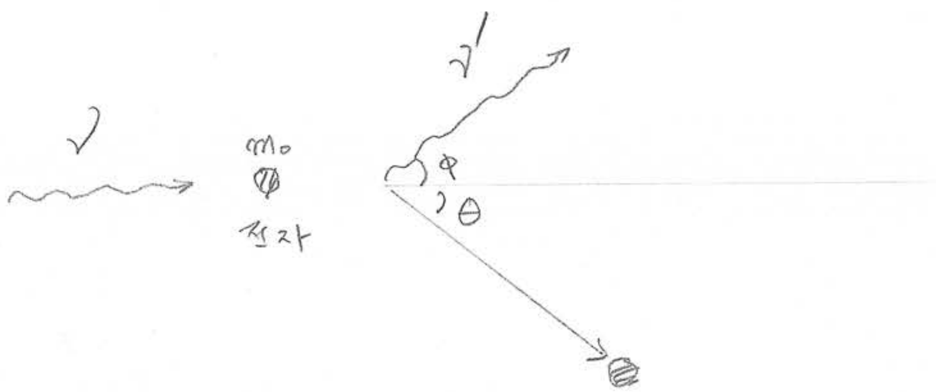
Since $eV \gg h\nu_0$,

$$eV \sim h\frac{c}{\lambda_{min}}$$

$$\boxed{\therefore \lambda_{min} = \frac{hc}{eV} = \frac{1.24 \times 10^{-6}\ (V m)}{V}}$$ ok

P~~81~~ 72 예제 : 생략

§ Compton 효과



$\nu' < \nu$: Compton 효과

$(\lambda' > \lambda)$ (1923년 Compton)

상대론 $E = \sqrt{p^2c^2 + m_0^2c^4}$

photon: $m_0 = 0$

$\Rightarrow E = p_p c = h\nu$

$\therefore p_p = \frac{h\nu}{c}$: photon의 운동량

p_e : electron의 충돌 후 운동량

x 방향: $\frac{h\nu}{c} = \frac{h\nu'}{c}\cos\phi + p_e \cos\theta$ — ①

$\frac{h\nu'}{c}\sin\phi = p_e \sin\theta$ — ②

① × c

$$P_e c \cos\theta = h\nu - h\nu' \cos\phi \quad - ③$$

② × c

$$P_e c \sin\theta = h\nu' \sin\phi \quad - ④$$

$③^2 + ④^2$

$$P_e^2 c^2 = (h\nu)^2 - 2(h\nu)(h\nu')\cos\phi + (h\nu')^2 \quad - ⑤$$

K : 전자의 충돌후 운동에너지

에너지 보존 법칙 :

$$\underbrace{K}_{\equiv mc^2 - m_0 c^2} = h\nu - h\nu' \quad - ⑥$$

E_e : 전자의 상대론적 에너지

$$\underbrace{E_e}_{\equiv mc^2} = K + m_0 c^2 = \sqrt{m_0^2 c^4 + P_e^2 c^2}$$

$$\Rightarrow \quad P_e^2 c^2 = K^2 + 2 m_0 c^2 K \quad - ⑦$$

⑥ → ⑦

$$P_e^2 c^2 = (h\nu)^2 - 2(h\nu)(h\nu') + (h\nu')^2$$

$$+ 2 m_0 c^2 (h\nu - h\nu') \quad - ⑧$$

From ⑤ & ⑧

$$2m_0c^2(h\nu - h\nu') = 2(h\nu)(h\nu')(1-\cos\phi) \quad — ⑨$$

$$⑨ \times \frac{1}{2h^2c^2}$$

$$\frac{m_0c}{h}\left(\frac{\nu}{c} - \frac{\nu'}{c}\right) = \frac{\nu}{c}\,\frac{\nu'}{c}\,(1-\cos\phi)$$

Since $\lambda\nu = c$,

$$\frac{m_0c}{h}\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) = \frac{1-\cos\phi}{\lambda\lambda'}$$

$$\Rightarrow \boxed{\lambda' - \lambda = \frac{h}{m_0c}(1-\cos\phi)}$$

note)

① $\lambda_c \equiv \dfrac{h}{m_0c}$: Compton wavelength

Ex) Electron

$$\lambda_c = 2.426 \times 10^{-12}\ \mathrm{m} \approx 2.4\ \mathrm{pm}$$

(Report) 본인의 compton wavelength를 구하라.

② $\phi = \pi$: $\lambda' - \lambda = 2\lambda_c$ $\Rightarrow$ p98 예제

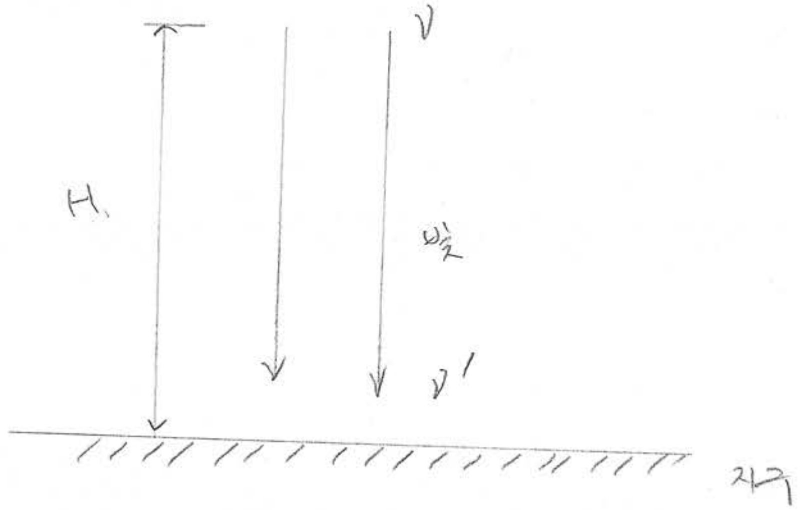
③ If $m_0 = \infty$, $\lambda' = \lambda$.

p94 § 광자와 중력

광자 $p = \frac{h\nu}{c}$

Since $m = \frac{p}{v}$, 중력장하에서 photon은

$\boxed{m = \frac{h\nu}{c^2}}$ 인 것처럼 행동한다.

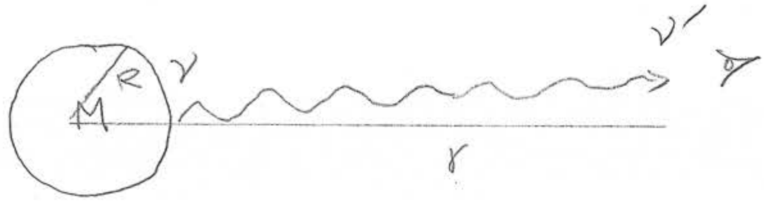


$$h\nu' = h\nu + \frac{h\nu}{c^2} g H$$

$$\boxed{\nu' = \nu\left(1 + \frac{gH}{c^2}\right) > \nu}$$

p~~96~~ 86 예 : 생략

§ Black hole



$G = 6.6726 \times 10^{-11}\ (Nm^2/kg^2)$

$$V = -\frac{GM}{r} m$$

For photon $m = \frac{h\nu}{c^2}$

$$\therefore \quad V = -\frac{GM}{r}\frac{h\nu}{c^2} = -\frac{GMh\nu}{c^2 r} \quad — ①$$

$$\therefore \quad h\nu - \frac{GMh\nu}{c^2 R} = h\nu' - \frac{GMh\nu'}{c^2 r}$$

If $r \gg 1$,

$$h\nu' = h\nu - \frac{GMh\nu}{c^2 R}$$

$$\boxed{\frac{\nu'}{\nu} = 1 - \frac{GM}{c^2 R} < 1}$$

red shift (적색 편이)

If $\frac{GM}{c^2 R} > 1$, $\frac{\nu'}{\nu} < 0 \Rightarrow$ impossible

So 빛이 별을 떠날수 없다. ⇒ Black hole

일반 상대론 condition for black hole

$$\frac{GM}{c^2 R} \geqq \frac{1}{2}$$

$$\boxed{R_s = \frac{2GM}{c^2}}$$

Schwarzschild radius

~~note~~ Report : 본인의 Schwarzschild radius 를 구하라

CH 3 입자의 파동성

§ De Broglie 물질파

1924년 De Broglie (과제 : Report)

운동하는 물체는 파동성을 갖는다.

$$\boxed{\lambda = \frac{h}{p} \qquad \nu = \frac{E}{h}}$$

"물질파"

If $\lambda \ll$ matter size, 입자성이 강함 (p 94 ~~147, 113~~ 문제)

(숙제)
본인의 100m 최고 기록을 이용하여 본인의 λ를 계산할것.

$\Psi(\vec{x}, t)$: 파동함수

(1) Schrödinger 방정식을 만족한다.

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[-\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + V \right] \Psi \qquad (\text{CH 5})$$

(2) $|\Psi|^2 = \Psi \Psi^*$

시간 t 에 공간 (x, y, z) 에서 물체를 발견할 확률

§ 파동의 수학적 표현

$$\boxed{y = A \cos 2\pi \left(\nu t - \frac{x}{\lambda} \right)}$$

A : 진폭 (Amplitude)

ν : 진동수 (frequency) $\left(\nu = \frac{1}{T} \quad T: \text{주기} \right)$

λ : 파장 (wave-length)

$v_p = \lambda \nu$ (파동의 위상속도 (phase velocity))

Definition

$k = \frac{2\pi}{\lambda}$ 파수 (wave number)

$\omega = 2\pi \nu$ 각진동수 (angular frequency)

$$\boxed{y = A \cos(\omega t - kx)}$$

$v_p = \frac{\omega}{k}$: 파동의 위상속도 (phase velocity)

$v_g = \frac{d\omega}{dk}$: 파동의 군속도 (group velocity)

(Ex) 물질파

$$K = \frac{2\pi}{\lambda} = \frac{2\pi}{h} p = \frac{2\pi}{h} m v = \frac{2\pi}{h} \frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\omega = 2\pi \nu = 2\pi \frac{E}{h} = 2\pi \frac{mc^2}{h} = \frac{2\pi m_0 c^2}{h\sqrt{1-\frac{v^2}{c^2}}}$$

$$v_p = \frac{\omega}{K} = \frac{c^2}{v} > c$$

$$v_g = \frac{d\omega}{dK} = \frac{\frac{d\omega}{dv}}{\frac{dK}{dv}} = v$$ (즉 v_g = 물체의 속도)

p ~~127~~ 103 문제

(Ex) Electromagnetic wave

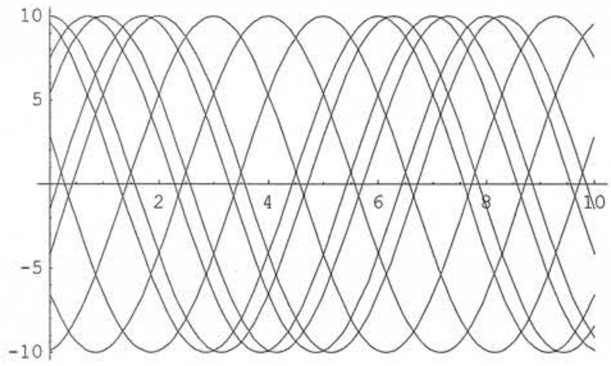
$$K = \frac{2\pi}{\lambda} \qquad \lambda\nu = c$$

$$\omega = 2\pi\nu$$

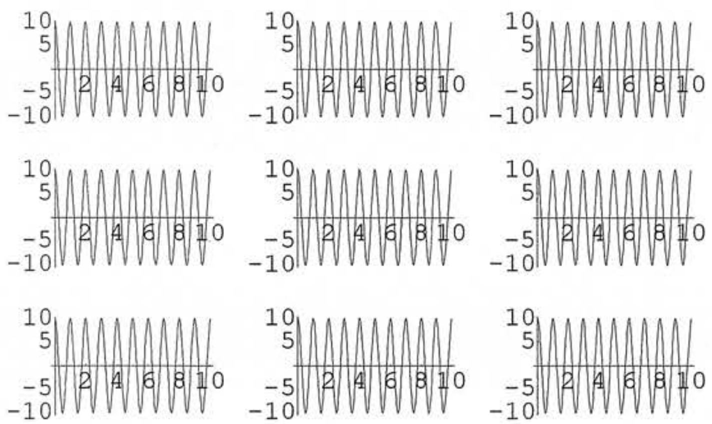
$$v_p = \frac{\omega}{k} = \lambda\nu = c$$

$$v_g = \frac{d\omega}{dk} = \frac{d\omega}{d\lambda}\left(\frac{dk}{d\lambda}\right)^{-1} = c$$

```
In[2]:= Needs["Graphics`Animation`"]

In[9]:= y[x_, t_] := 10 Cos[ (t - x)]

In[10]:= Plot[Evaluate[Table[y[x, t], {t, 0, 8}]], {x, 0, 10}]
```

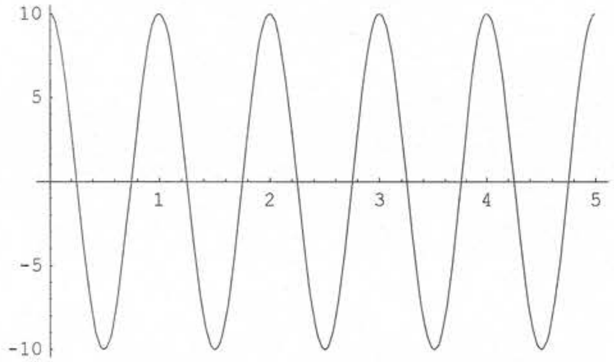


```
Out[10]= - Graphics -

In[11]:= Table[Plot[Evaluate[y[x, t]], {x, 0, 10},
            PlotRange -> All, DisplayFunction -> Identity], {t, 0, 8}] //
          Short

Out[11]//Short=
       {- Graphics -, - Graphics -, <<5>>, - Graphics -, - Graphics -}

In[12]:= Show[GraphicsArray[Partition[%6, 3]]]
```

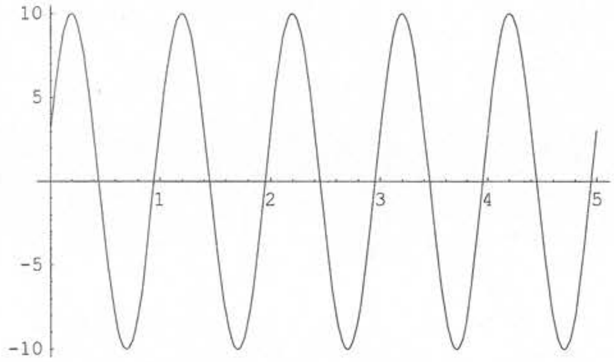


```
Out[12]= - GraphicsArray -

In[13]:= ShowAnimation[GraphicsArray[Partition[%6, 3]]]

Out[13]= ShowAnimation[- GraphicsArray -]
```

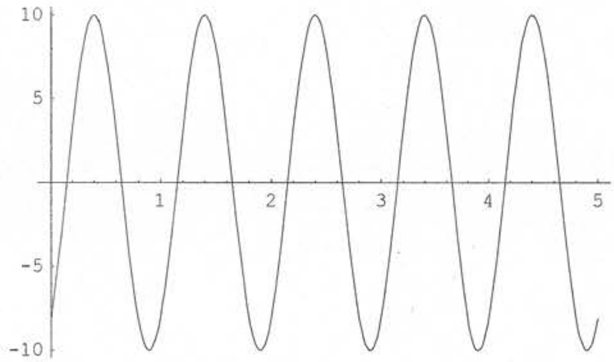
```
In[1]:= Plot[10 Cos[2 Pi (1 - x)], {x, 0, 5}]
```



```
Out[1]= - Graphics -

In[3]:= Plot[10 Cos[2 Pi (1.2 - x)], {x, 0, 5}]
```

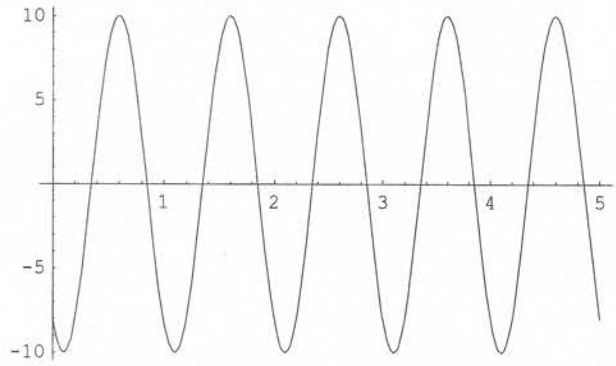


```
Out[3]= - Graphics -

In[4]:= Plot[10 Cos[2 Pi (1.4 - x)], {x, 0, 5}]
```

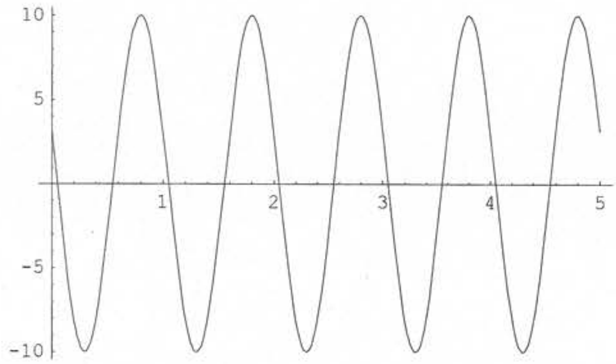


```
Out[4]= - Graphics -
```

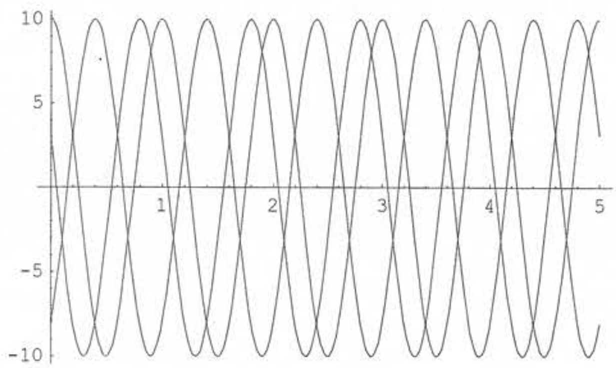
```
In[5]:= Plot[10 Cos[2 Pi (1.6 - x)], {x, 0, 5}]
```




```
Out[5]= - Graphics -

In[6]:= Plot[10 Cos[2 Pi (1.8 - x)], {x, 0, 5}]
```



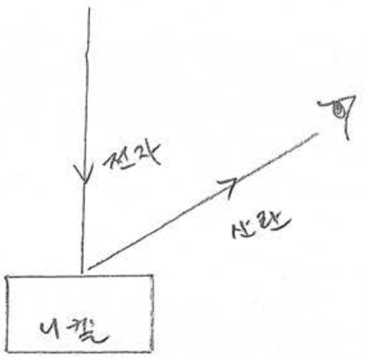

```
Out[6]= - Graphics -

In[9]:= Show[%1, %4, %6]
```



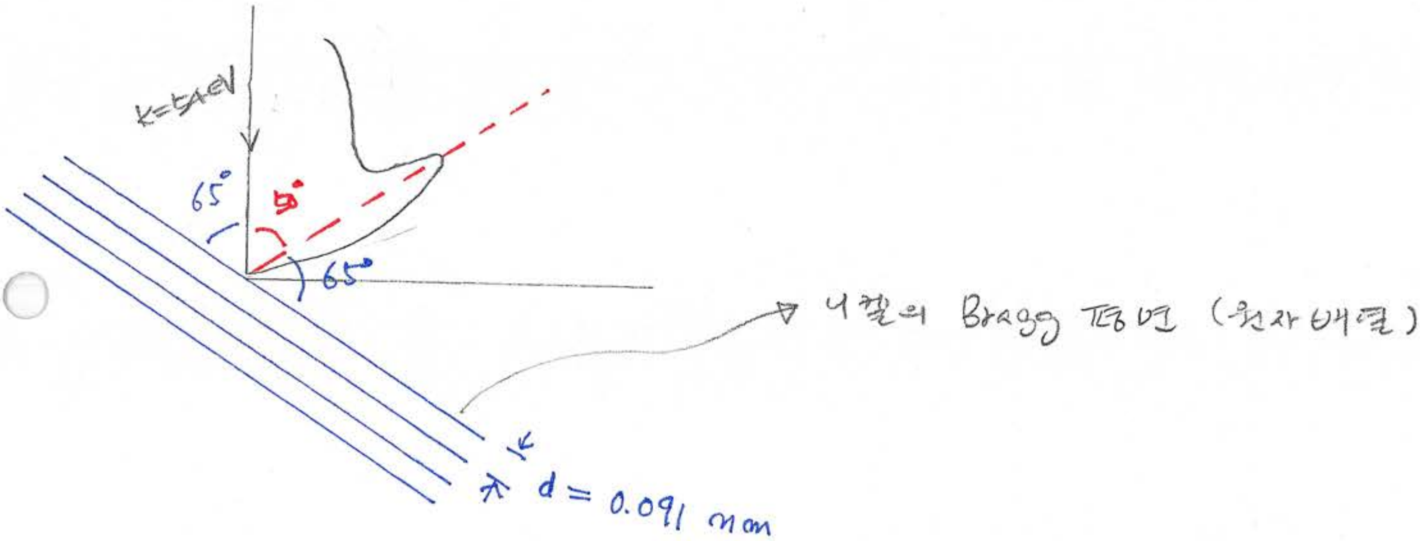

```
Out[9]= - Graphics -
```

§ 입자 회절

1927년 Davisson and Germer : 전자 산란 실험



Polar Plot



파동에서 Bragg 회절 공식

θ: 산란각도

$2d \sin\theta = n\lambda$ (보강간섭 조건)

$(n = 1, 2, \cdots)$

$\theta = 65°$

If $n=1$, $\lambda = 2d\sin\theta = 0.165$ nm

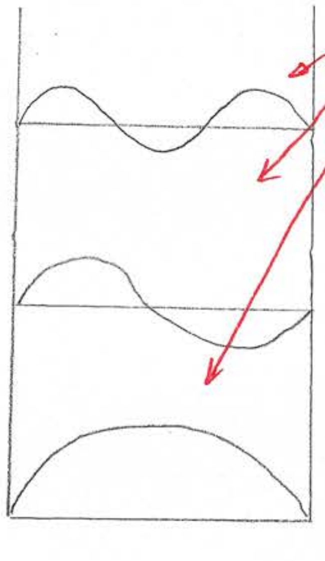
Since $K = 54$ eV, $mv = \sqrt{2mK} = 4.0 \times 10^{-24}$ kg m/sec

$\lambda = \dfrac{h}{mv} = 0.166$ nm

전자가 회절한다 !!

◎ 상자속의 입자 (에너지 양자화)

가능한 matter wave = 정상파



$\lambda = \frac{2}{3}L$

$\lambda = L$

$\lambda = 2L$

L

$$\lambda_n = \frac{2L}{n} = \frac{h}{p} \qquad (n = 1, 2, \cdots)$$

$$p_n = \frac{nh}{2L}$$

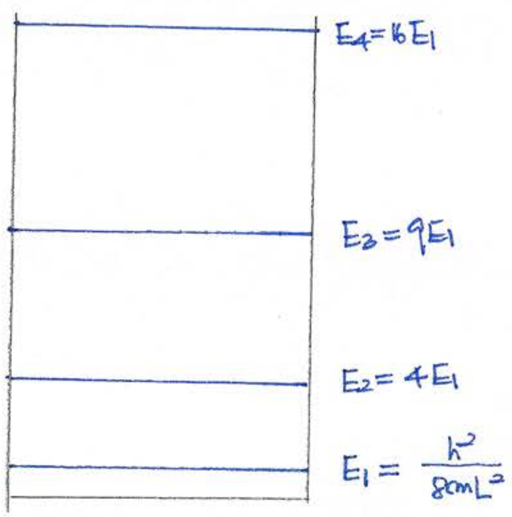
$$\boxed{E_n = \frac{p_n^2}{2m} = \frac{n^2 h^2}{8mL^2} \qquad n = 1, 2, \cdots}$$

에너지 준위

상자속 입자의 가능한

energy 상태

n : 양자수



~~[illegible]~~ 1. 7

p~~[illegible]~~

문제

5 불확정성 원리

$$\boxed{\begin{array}{c} \Delta x \, \Delta p \geq \frac{\hbar}{2} \\[1em] \Delta E \, \Delta t \geq \frac{\hbar}{2} \end{array}}$$ ← 암기

1927년 Heisemberg (고사 발표)

⇒ 어떤 물체의 운동량과 위치를 동시에 정확히 측정할수 없다.

112
p~~13~~ 문제

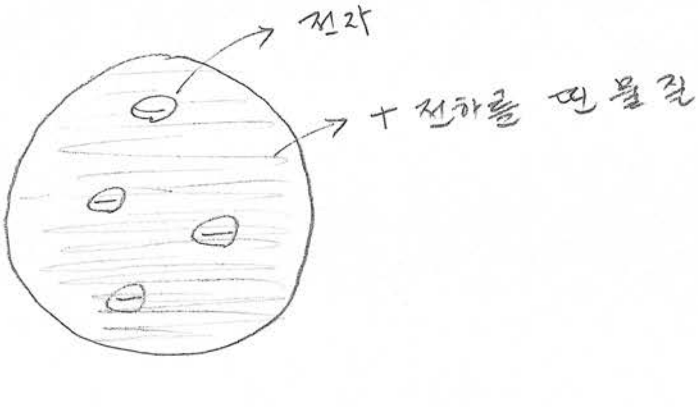
p~~1~~ 문제
115

p116 문제

CH 4. 원자의 구조

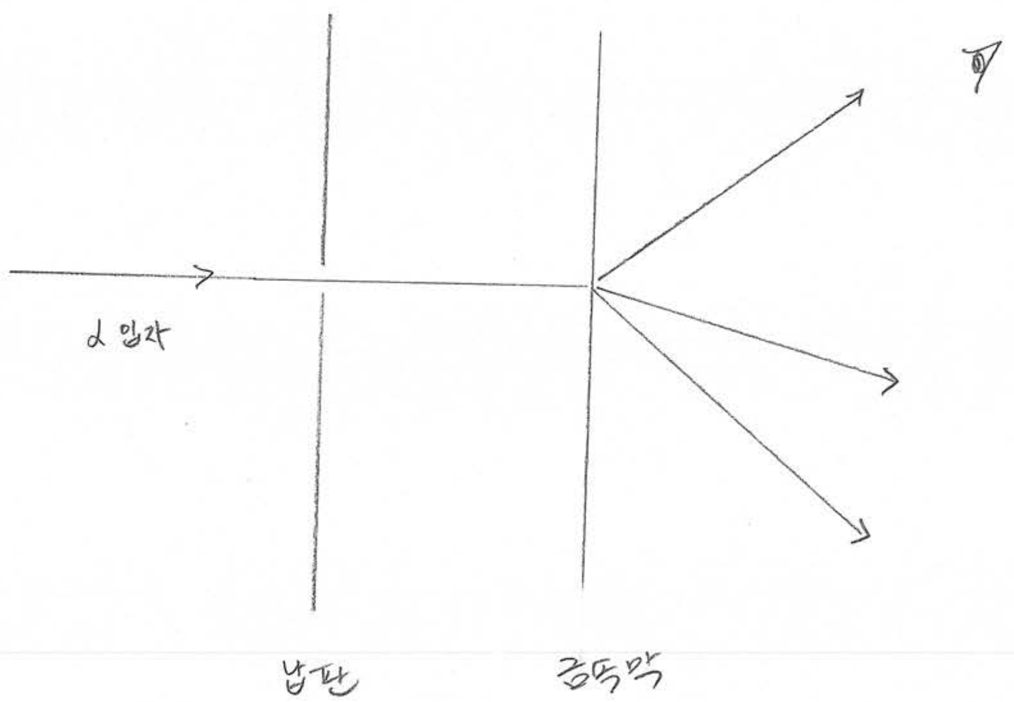
§ Rutherford 의 원자모형

1898년 J. J. Thomson

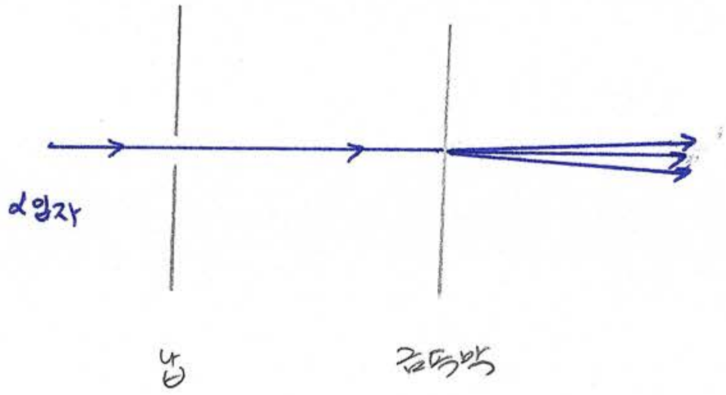


1911년 Geiger and Marsden : α 입자 산란실험

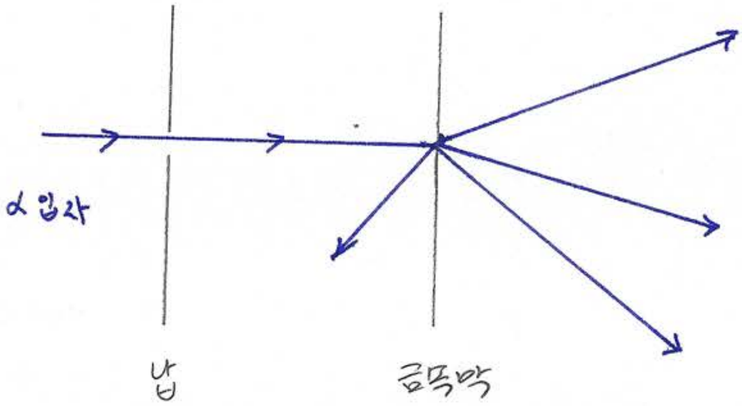
α 입자 : Helium 원자핵



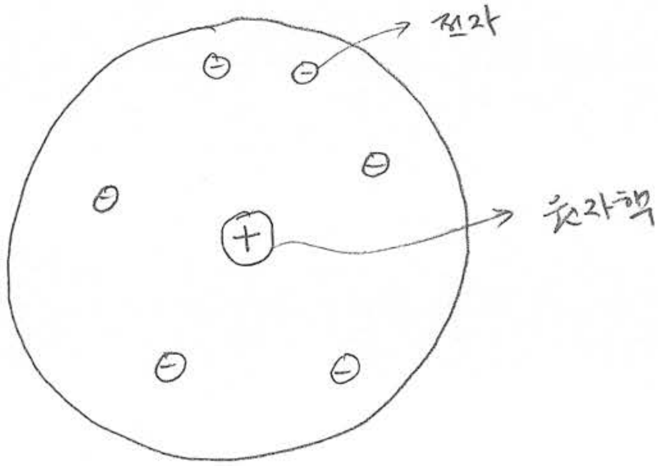
원자의 Thomson 모형이 맞다면 굴절각 $\simeq 1^\circ$



실제 실험결과



Rutherford 모형 (1911년) : 원자에는 원자핵이 존재한다



§ α 입자 산란

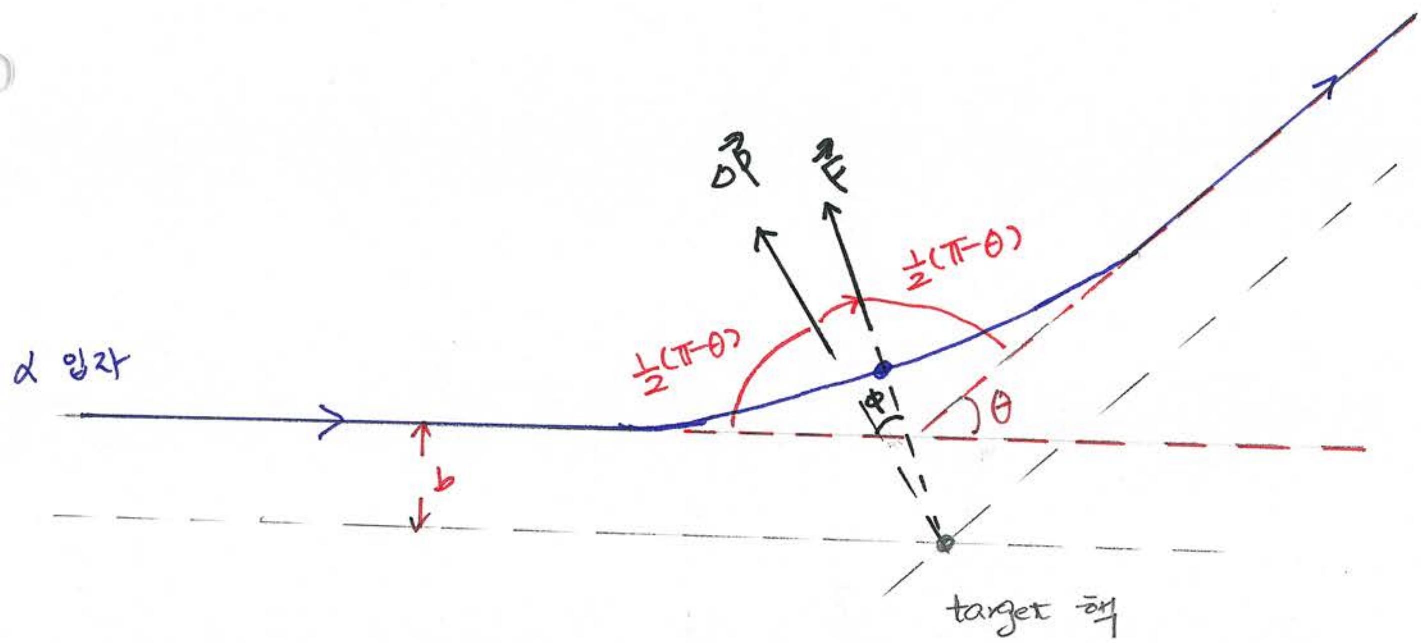


그림 1

b: impact parameter (충돌변수)

θ: scattering angle (산란각)

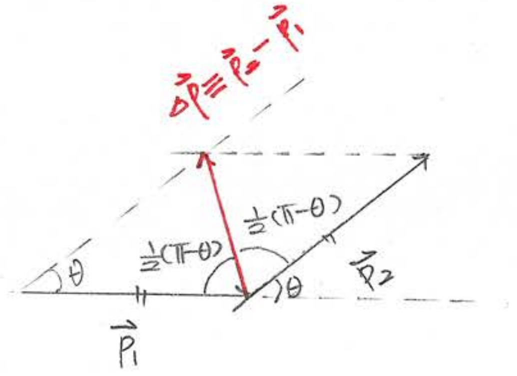


그림 2

$$|\vec{p}_1| = |\vec{p}_2| = mv$$

m: α 입자의 질량

v: α 입자의 속력

그림 2 로부터

$$\frac{\Delta p}{\sin\theta} = \frac{mv}{\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)} = \frac{mv}{\cos\frac{\theta}{2}}$$

$\Rightarrow$

$$\Delta p = 2mv \sin\frac{\theta}{2} \qquad -①$$

일반 물리 ; 충격량

$$\boxed{\vec{J} \equiv \int \vec{F}\, dt = \Delta\vec{p}}$$

$\therefore$

$$2mv \sin\frac{\theta}{2} = \int_{-\infty}^{\infty} F\cos\phi \; dt \qquad -②$$

$$\int_{-\infty}^{\infty} F\cos\phi\, dt = \int_{-\frac{\pi-\theta}{2}}^{\frac{\pi-\theta}{2}} F\cos\phi \frac{dt}{d\phi}\, d\phi \qquad -③$$

③ → ②

$$2mv \sin\frac{\theta}{2} = \int_{-\frac{\pi-\theta}{2}}^{\frac{\pi-\theta}{2}} F\cos\phi \frac{dt}{d\phi}\, d\phi \qquad -④$$

$$\frac{d\phi}{dt} = \omega \quad (\alpha \text{ 입자의 각속도})$$

α 입자의 각운동량 보존 법칙.

$$\vec{L} \equiv \vec{r} \times \vec{p} = I\vec{\omega} \quad (I: \text{관성능률})$$

$$\Rightarrow m\omega r^2 = mvb \qquad -⑤$$

r: target 핵과 α 입자 사이의 거리

$$\therefore \quad \omega = \frac{vb}{r^2} \qquad -⑥$$

$$\therefore \quad \frac{dt}{d\phi} = \left(\frac{d\phi}{dt}\right)^{-1} = \frac{r^2}{vb} \qquad -⑦$$

⑦ → ④

$$2mv^2 b \sin\frac{\theta}{2} = \int_{-\frac{\pi-\theta}{2}}^{\frac{\pi-\theta}{2}} F r^2 \cos\phi \, d\phi \qquad -⑧$$

α 입자의 전하량: $2e$

핵의 전하량 : Ze (Z: 원자번호)

$$F = \frac{1}{4\pi\epsilon_0} \frac{2Ze^2}{r^2} \qquad -⑨$$

⑨ → ⑧

$b \Leftrightarrow \theta$ relation !!

$$\cot\frac{\theta}{2} = \frac{2\pi\epsilon_0 m v^2}{Ze^2} b \qquad -⑩$$

If $b=0$

$$\cos\frac{\theta}{2}=0, \quad \theta=\pi \quad ; \quad \text{backward scattering}$$

If $b=\infty$

$$\sin\frac{\theta}{2}=0, \quad \theta=0 \quad ; \quad \text{forward scattering}$$

Let

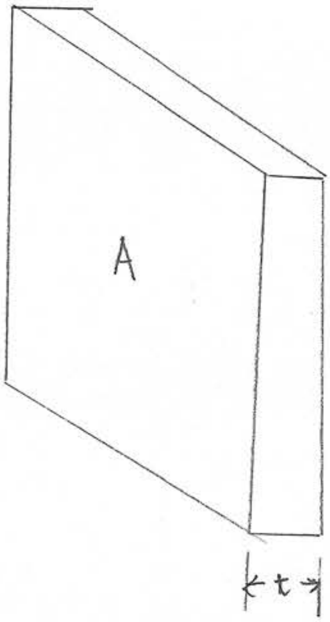
$$K=\frac{1}{2}mv^2 \quad — ⑪ \quad ; \text{ α-입자의 운동에너지}$$

$$\boxed{\cot\frac{\theta}{2}=\frac{4\pi\epsilon_0 K}{Ze^2}\,b} \quad — ⑫$$

$0 <$ impact parameter $\leq b$ $\longleftrightarrow$ $\theta \leq$ scattering angle

scattering cross section $\leq \pi b^2$

n : 단위 체적당 원자수



total 원자수 $= nAt$

total cross section $= nAt\pi b^2$

$$f \equiv \frac{\theta \text{ 이상의 각도로 산란되는 } \alpha \text{ 입자의 갯수}}{\text{입사하는 } \alpha\text{-입자의 전체 갯수}}$$

$$= \frac{nAt\pi b^2}{A}$$

$$= nt\pi b^2 \qquad -⑬$$

⑫ → ⑬

$$f = \pi n t \left(\frac{Ze^2}{4\pi\epsilon_0 K}\right)^2 \cot^2\frac{\theta}{2} \qquad -⑭$$

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⇒ p 예제

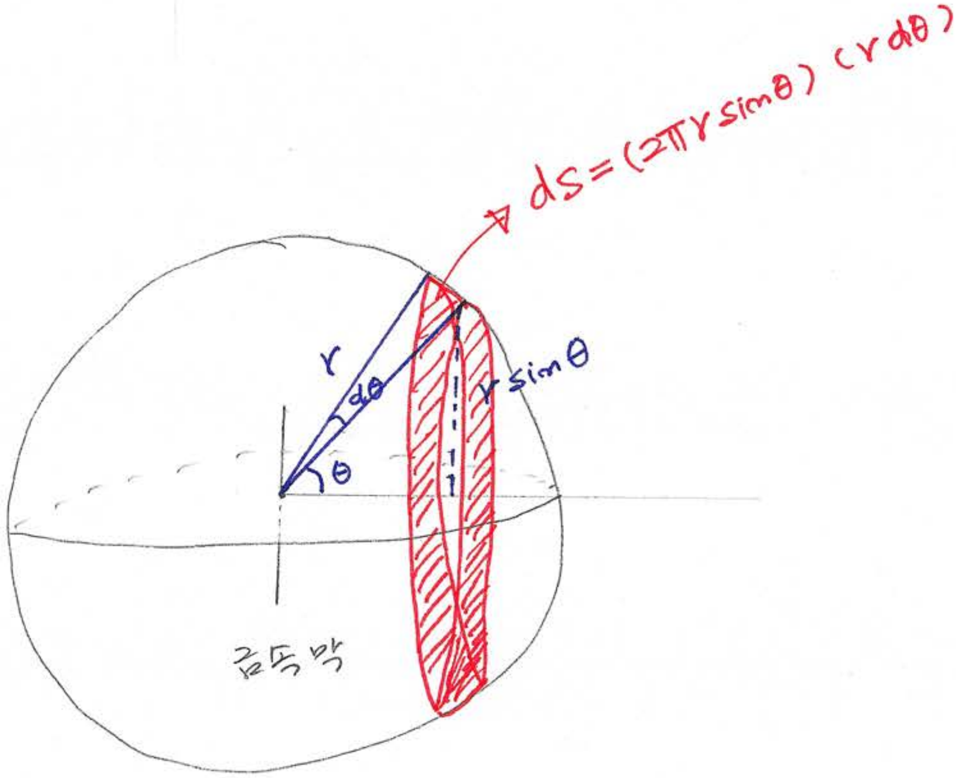
$$df \equiv \frac{\text{산락각이 } \theta \text{ 에서 } \theta + d\theta \text{ 인 } \alpha\text{-입자의 갯수}}{\text{입사하는 } \alpha\text{-입자의 전체갯수}}$$

$$= -\pi n t \left(\frac{Ze^2}{4\pi\epsilon_0 K} \right)^2 \cot\frac{\theta}{2} \, \mathrm{cosec}^2\frac{\theta}{2} \, d\theta \qquad \text{— ⑮}$$

meaning of $-$

; θ가 증가 할 때 f가 감소함

실제 실험



$$dS = 2\pi r^2 \sin\theta \, d\theta = 4\pi r^2 \sin\frac{\theta}{2} \cos\frac{\theta}{2} \, d\theta \qquad \text{— ⑯}$$

N_i: 입사된 total α-입자수

$N_i\,df$: scattering angle 이 θ 와 θ+dθ 사이로

산란된 α-입자 수

$N(\theta)$: Scattering angle 이 θ 와 θ+dθ 사이이고,

단위 면적당 산란된 α-입자 수

$$\boxed{N(\theta) = \frac{N_i |df|}{dS} = \frac{N_i n t Z^2 e^4}{(8\pi\epsilon_0)^2 r^2 K^2} \frac{1}{\sin^4 \frac{\theta}{2}}}$$

Rutherford 산란 공식

⇒ Geiger & Marsden 실험 결과와 일치

<u>모든 원자는 "원자핵"을 가진다</u>

§ 원자핵의 크기

핵의 반지름 $\leq r_0$

r_0: α 입자의 최근접거리

r_0에서 α 입자의 속도 $= 0 \Rightarrow$ no 운동 에너지

$$K = \frac{1}{4\pi\epsilon_0} \frac{2Ze^2}{r_0}$$

$$\Rightarrow \quad r_0 = \frac{2Ze^2}{4\pi\epsilon_0 K}$$

(문제 실험)

금: $Z = 79$

$$K = 7.7\ \text{MeV} = 1.2 \times 10^{-12}\ (\text{J})$$

$$r_0 \approx 3 \times 10^{-14}\ \text{m}$$

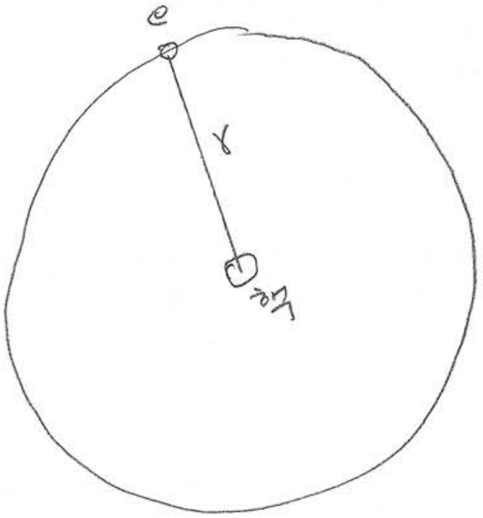
핵의 반지름 $\leq 3 \times 10^{-14}$ m

현대실험: 핵의 반지름 $\sim 10^{-15}$ m

§ 전자의 궤도

전자가 원자핵 주위에서 안정하기 위해서는 운동하여야 한다

원운동 : 수소전자



$$F = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2} = m\frac{v^2}{r}$$ (m : 전자의 질량)

$$\Rightarrow \quad v = \frac{e}{\sqrt{4\pi\epsilon_0 m r}} \quad -①$$

$$K = \frac{1}{2}mv^2 = \frac{e^2}{8\pi\epsilon_0 r}$$

$$V = -\frac{e^2}{4\pi\epsilon_0 r}$$

$$E = K + V = -\frac{e^2}{8\pi\epsilon_0 r} \quad -②$$

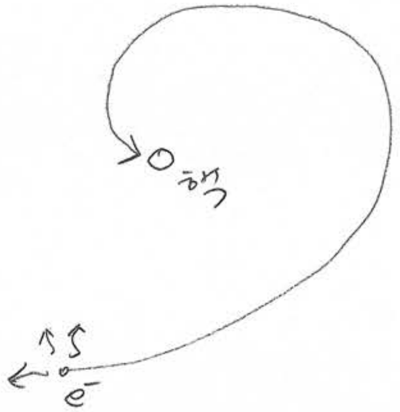
meaning of −

⇒ p156 문제

: 전자가 핵에 구속되어 있다.

전자기학

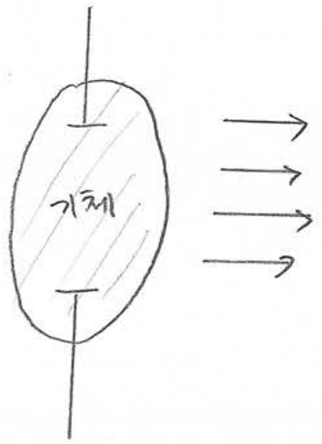
⇒ 모든 가속하는 전하는 전자기파를 방출한다.



⇒ 새로운 이론이 필요

⇒ 양자역학

§ 원자 spectrum



빛 (λ).

전기방전

수소 기체

$\frac{1}{\lambda} = R\left(\frac{1}{1^2} - \frac{1}{n^2}\right)$ $n = 2, 3, 4 \cdots$ (자외선) Lyman 계열 (자외선)

$\frac{1}{\lambda} = R\left(\frac{1}{2^2} - \frac{1}{n^2}\right)$ $n = 3, 4, 5, \cdots$ (가시광선) Balmer (가시광선) (1885)

$\frac{1}{\lambda} = R\left(\frac{1}{3^2} - \frac{1}{n^2}\right)$ $n = 4, 5, 6 \cdots$ (적외선) Paschen

$\frac{1}{\lambda} = R\left(\frac{1}{4^2} - \frac{1}{n^2}\right)$ $n = 5, 6, 7, \cdots$ Brackett

$\frac{1}{\lambda} = R\left(\frac{1}{5^2} - \frac{1}{n^2}\right)$ $n = 6, 7, 8, \cdots$ Pfund

$R = 1.097 \times 10^7 \ (m^{-1})$ Rydberg constant

◎ Bohr의 원자 모형

Bohr의 가정 (1913년)

원자 내 전자의 궤도가 전자의 파장의 정수배이다

$$\boxed{\begin{gathered} 2\pi r_n = n\lambda \\ \lambda = \frac{h}{mv} \\ n = 1, 2, 3, \cdots \end{gathered}}$$

$$2\pi r_n = \frac{nh}{mv} \quad \Leftarrow \quad \left(v = \frac{e}{\sqrt{4\pi\epsilon_0 m r}} \right)$$

$$\Rightarrow \quad 2\pi r_n = \frac{nh}{e}\sqrt{\frac{4\pi\epsilon_0 r_n}{m}}$$

$$\Rightarrow \quad \underline{r_n = a_0 n^2}$$

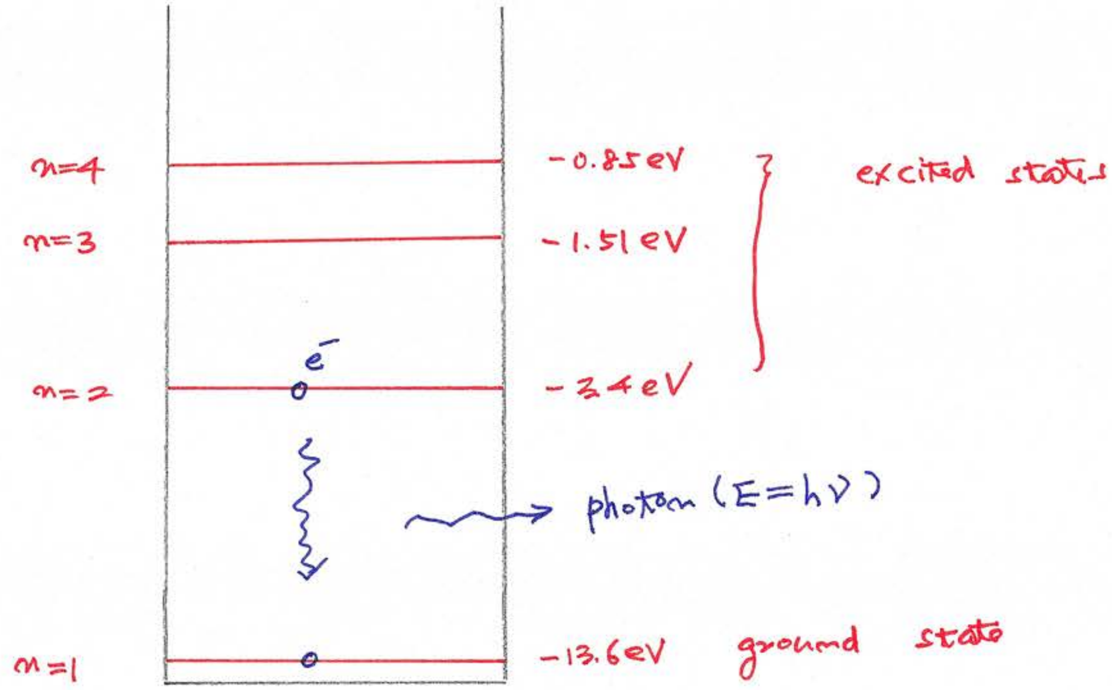
$$a_0 = \frac{\epsilon_0 h^2}{\pi m e^2} = 5.292 \times 10^{-11}\ \text{m}$$: Bohr radius

From $E = -\dfrac{e^2}{8\pi\epsilon_0 r}$,

$$\underline{E_n = -\frac{me^4}{8\epsilon_0^2 h^2}\frac{1}{n^2}}$$: 수소원자의 energy state

$n=1$: $E_1 = -13.6\,\text{eV}$ ground state

$n \neq 1$: Excited state



transition from $n = n_i$ to $n = n_f$

$$E_i - E_f = \frac{me^4}{8\epsilon_0^2 h^2}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) = h\nu = h\frac{c}{\lambda}$$

$$\frac{1}{\lambda} = \frac{me^4}{8\epsilon_0^2 h^3 c}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

if $n_f = 1$: Lyman

If $n_f = 2$: Balmer

If $n_f = 3$: Paschen

If $n_f = 4$: Brackett

If $n_f = 5$: Pfund

Rydberg constant

$$R = \frac{me^4}{8\epsilon_0^2 h^3 c} = 1.097 \times 10^7 \ (m^{-1})$$

CH 5 양자역학

8. 파동함수 (Wave Function)

$\Psi(t, x, y, z)$: 파동함수

$$|\Psi(t,x,y,z)|^2 \equiv \Psi^*(t,x,y,z)\,\Psi(t,x,y,z)$$

: 시간 t에 공간 (x, y, z)에서 입자를 발견할 확률

$$\Rightarrow \int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dz\ |\Psi(t,x,y,z)|^2 = 1$$

normalization condition

(Ex) 입자를 $x=x_1$ 과 $x=x_2$ 사이에서 발견할 확률은?

$$\int_{x_1}^{x_2}dx\ |\Psi|^2 = P$$

§ time-dependent Schrödinger Equation

파동의 표현

$$\Psi = \begin{cases} A\cos 2\pi\left(\nu t - \dfrac{x}{\lambda}\right) \\ \text{or} \\ A\sin 2\pi\left(\nu t - \dfrac{x}{\lambda}\right) \end{cases}$$

See CH3

같이 표현

$$\Psi = A e^{-i2\pi\left(\nu t - \frac{x}{\lambda}\right)}$$

$$\left(e^{i\theta} \equiv \cos\theta + i\sin\theta\right)$$

물질파

$$\begin{pmatrix} P = \dfrac{h}{\lambda} \\ E = h\nu \end{pmatrix}$$

$$\Psi = A e^{-\frac{i}{\hbar}(Et - px)} \quad - ① \qquad \left(\hbar \equiv \frac{h}{2\pi}\right)$$

$$\left.\begin{aligned} \frac{\partial \Psi}{\partial t} &= \frac{-i}{\hbar} E\,\Psi \\ \frac{\partial^2 \Psi}{\partial x^2} &= -\frac{p^2}{\hbar^2}\,\Psi \end{aligned}\right\} \quad - ②$$

$$E = T + V = \frac{p^2}{2m} + V,$$

$$\frac{\partial \Psi}{\partial t} = -\frac{i}{\hbar}\left(\frac{p^2}{2m} + V\right)\Psi$$

$$= -\frac{i}{\hbar}\frac{1}{2m}\underline{p^2 \Psi} - \frac{i}{\hbar} V \Psi$$

$$-\hbar^2 \frac{\partial^2 \Psi}{\partial x^2}$$

$$= \frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar} V \Psi \qquad \| \, i\hbar$$

$$\underline{i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V(t, x)\Psi}$$

1차원 time-dependent Schrödinger Equation

⇒ 3차원으로 확장

$$\boxed{\begin{aligned} i\hbar \frac{\partial \Psi}{\partial t} &= -\frac{\hbar^2}{2m}\nabla^2 \Psi + V(t, \vec{r})\Psi \\ \nabla^2 &\equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \ : \text{Laplacian} \end{aligned}}$$

3차원 time-dependent Schrödinger Equation

§ time-independent Schrödinger Equation

Consider 1-dim time-dep Schrödinger Eq :

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V(x,t)\,\Psi$$

If $V = V(x)$, put $\Psi = \phi(x)\,T(t)$

$$i\hbar\, \phi(x) \frac{dT}{dt} = -\frac{\hbar^2}{2m} T \frac{d^2\phi}{dx^2} + V(x)\, T\, \phi \qquad \| \times \frac{1}{T\phi}$$

$$\underbrace{i\hbar \frac{1}{T} \frac{dT}{dt}}_{\text{function of } t} = \underbrace{-\frac{\hbar^2}{2m} \frac{1}{\phi} \frac{d^2\phi}{dx^2} + V(x)}_{\text{function of } x} \equiv E$$

$$i\hbar \frac{dT}{dt} = E$$

$$\therefore\ T = e^{-\frac{i}{\hbar}Et}$$

$$\underline{-\frac{\hbar^2}{2m} \frac{d^2\phi}{dx^2} + V(x)\,\phi = E\,\phi}$$

1차원 time-independent Schrödinger Eq

⇒ 3차원으로 확장

ϕ : eigenfunction

$$\underline{-\frac{\hbar^2}{2m} \nabla^2\phi + V(x,y,z)\,\phi = E\,\phi}$$

3차원 time-independent Schrödinger Eq

§ 기대치 (Expectation Value)

주사위

n_i : 나올 가능한 data

P_i : n_i 가 나올 확률

n_i	P_i
1	$\frac{1}{6}$
2	$\frac{1}{6}$
⋮	⋮
6	$\frac{1}{6}$

$\langle n \rangle$: Expectation value (기대치)

한번에 얻을 평균 값

$$\langle n \rangle \equiv \sum_i n_i P_i = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + \cdots + 6 \times \frac{1}{6} = 3.5$$

양자 역학

$$\langle x \rangle = \int_{-\infty}^{\infty} dx \; x \, |\Psi|^2$$

$$\langle f(x) \rangle = \int_{-\infty}^{\infty} dx \; f(x) \, |\Psi|^2$$

p215 예제

§ Particle in a Box

$$V(x) = \begin{cases} 0 & 0 < x < L \\ \infty & \text{elsewhere} \end{cases}$$

time-indep Schrödinger Eq

$$-\frac{\hbar^2}{2m}\frac{d^2\phi}{dx^2} = E\phi$$

$$\Rightarrow \quad \phi = A \sin\frac{\sqrt{2mE}}{\hbar}x + B\cos\frac{\sqrt{2mE}}{\hbar}x$$

$\phi(0) = \phi(L) = 0$: Box 밖에 존재할 확률은 zero 이다.

$$\Rightarrow \quad B = 0$$

$$\frac{\sqrt{2mE}}{\hbar}L = n\pi \qquad (n = 1, 2, 3, \cdots)$$

$$\underline{E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}}$$

에너지 양자화

$$\phi_n(x) = A \sin\frac{\sqrt{2mE_n}}{\hbar}x = A \sin\frac{n\pi x}{L}$$

$$\int_0^L |\phi_n(x)|^2 dx = 1 \quad \Rightarrow \quad A = \sqrt{\frac{2}{L}}$$

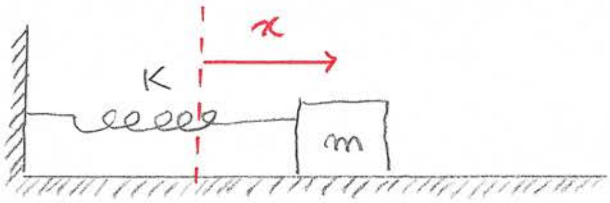
$$\Rightarrow \quad \underline{\phi_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}}$$

eigenfunction

P225, 226

문제

§ 조화 진동자



〈고전역학〉

$$F = -kx = m\frac{d^2x}{dt^2}$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

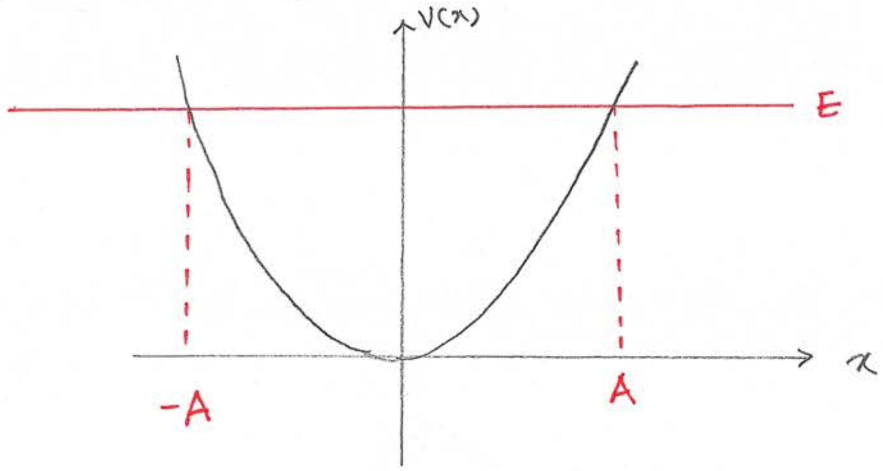
$$x = A\cos(\omega t + \phi)$$

A : amplitude

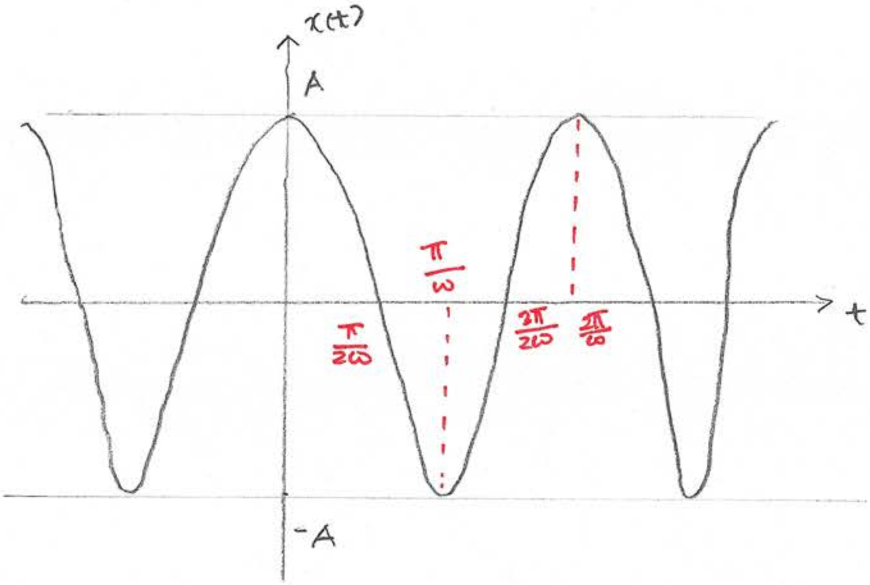
$\omega = \sqrt{\frac{k}{m}}$: angular frequency

ϕ : phase constant

potential energy ; $V(x) = \frac{1}{2}kx^2$



$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 = \text{const}$$



〈양자역학〉

Time-independent Schrödinger Eq.

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right]\phi = E\phi$$

$$-\frac{\hbar^2}{2m}\frac{d^2\phi}{dx^2}+\frac{1}{2}m\omega^2x^2\phi = E\phi \quad \Big| \times \left(-\frac{2m}{\hbar^2}\right)$$

$$\frac{d^2\phi}{dx^2}+(\beta-\alpha^2x^2)\phi = 0$$

$$\alpha = \frac{m\omega}{\hbar}, \quad \beta = \frac{2mE}{\hbar^2} \quad \text{— ①}$$

Change of variable

$$q = \sqrt{\alpha}\, x \quad \text{— ②}$$

Then

$$\frac{d^2\phi}{dq^2}+(\epsilon - q^2)\phi = 0 \quad \text{— ③}$$

$$\epsilon = \frac{\beta}{\alpha} = \frac{2E}{\hbar\omega} \quad \text{— ④}$$

Let

$$\phi(q) = e^{-\frac{q^2}{2}} H(q) \qquad - ⑤$$

⑤ → ③

$$\underline{H''(q) - 2q H'(q) + (\epsilon - 1) H(q) = 0} \qquad - ⑥$$

Hermite differential Eq.

Eq. ⑥ can be solved by series solution

$$\left.\begin{aligned} H(q) &= \sum_{k=0}^{\infty} a_k q^k \\ H'(q) &= \sum_{k=1}^{\infty} k a_k q^{k-1} \end{aligned}\right\} \qquad - ⑦$$

$$H''(q) = \sum_{k=2}^{\infty} k(k-1) a_k q^{k-2}$$

⑦ → ⑥

$$\underbrace{[2a_2 + (\epsilon-1) a_0]}_{=0} + \sum_{k=1}^{\infty} \underbrace{\left[(k+2)(k+1) a_{k+2} - \{2k - (\epsilon-1)\} a_k\right]}_{=0} q^k = 0$$

$$\left.\begin{aligned} a_2 &= -\frac{\epsilon - 1}{2} a_0 \\ a_{k+2} &= -\frac{\epsilon - 1 - 2k}{(k+2)(k+1)} a_k \end{aligned}\right\} \qquad - ⑧$$

Solving ② we get

$$a_2 = -\frac{\epsilon-1}{2!}a_0$$

$$a_3 = -\frac{(\epsilon-3)}{3!}a_1$$

$$a_4 = \frac{(\epsilon-5)(\epsilon-1)}{4!}a_0$$

$$a_5 = \frac{(\epsilon-7)(\epsilon-3)}{5!}a_1$$

$$a_6 = -\frac{(\epsilon-9)(\epsilon-5)(\epsilon-1)}{6!}a_0$$

$$a_7 = -\frac{(\epsilon-11)(\epsilon-7)(\epsilon-3)}{7!}a_1$$

$$\vdots$$

$$a_{2k} = (-1)^k \frac{[\epsilon-(4k-3)][\epsilon-(4k-7)]\cdots(\epsilon-5)(\epsilon-1)}{(2k)!}a_0$$

$$a_{2k+1} = (-1)^k \frac{[\epsilon-(4k-1)][\epsilon-(4k-5)]\cdots(\epsilon-7)(\epsilon-3)}{(2k+1)!}a_1$$

Thus $H(q)$ becomes

$$H(q) = a_0\left[1 - \frac{\epsilon-1}{2!}q^2 + \frac{(\epsilon-5)(\epsilon-1)}{4!}q^4 + \cdots\right]$$

$$+a_1\left[q - \frac{\epsilon-3}{3!}q^3 + \frac{(\epsilon-7)(\epsilon-3)}{5!}q^5 + \cdots\right] \quad -④$$

ⓐ → ⓔ

$$\phi(q) = e^{-\frac{q^2}{2}} H(q)$$

$$= e^{-\frac{q^2}{2}} \left[\begin{array}{l} a_0 \left[1 - \frac{\epsilon - 1}{2!} q^2 + \frac{(\epsilon-5)(\epsilon-1)}{4!} q^4 + \cdots \right] \\ + a_1 \left[q - \frac{\epsilon-3}{3!} q^3 + \frac{(\epsilon-7)(\epsilon-3)}{5!} q^5 + \cdots \right] \end{array} \right] \quad -⑲$$

From $\int_{-\infty}^{\infty} dx \, \phi^*(x) \, \phi(x) = 1$, the series must be terminated.

To terminate the series

① $\epsilon = 1 \equiv \frac{2E}{\hbar\omega}$ $\qquad E_0 = \frac{1}{2}\hbar\omega$, $a_1 = 0$

$$\phi_0 = C_0 \, e^{-\frac{q^2}{2}} = C_0 \, e^{-\frac{m\omega}{2\hbar} x^2}$$

② $\epsilon = 3 \equiv \frac{2E}{\hbar\omega}$ $\qquad E_1 = \frac{3}{2}\hbar\omega$, $a_0 = 0$

$$\phi_1 = C_1 \, q \, e^{-\frac{q^2}{2}} = C_1 \sqrt{\frac{m\omega}{\hbar}} \, x \, e^{-\frac{m\omega}{2\hbar} x^2}$$

③ $\epsilon = 5 = \frac{2E}{\hbar\omega}$ $\qquad E_2 = \frac{5}{2}\hbar\omega$, $a_1 = 0$

$$\phi_2 = C_2 \left(1 - 2q^2\right) e^{-\frac{q^2}{2}} = C_2 \left(1 - \frac{2m\omega}{\hbar} x^2\right) e^{-\frac{m\omega}{2\hbar} x^2}$$

④ $\epsilon = 7 = \frac{2E}{\hbar\omega}$ $\qquad E_3 = \frac{7}{2}\hbar\omega$ $\qquad a_0 = 0$

$$\phi_3 = C_3 \left(q - \frac{2}{3} q^3\right) e^{-\frac{q^2}{2}} = C_3 \left(\sqrt{\frac{m\omega}{\hbar}} x - \frac{2}{3} \left(\sqrt{\frac{m\omega}{\hbar}} x \right)^3 \right) e^{-\frac{m\omega}{2\hbar} x^2}$$

C_n should be determined from a condition

$$\int_{-\infty}^{\infty} dx \, \phi_n^*(x) \, \phi_n(x) = 1$$

Summary

$$E_n = \left(n + \frac{1}{2}\right) \hbar \omega \qquad (n = 0, 1, 2, \cdots)$$

$$\phi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right) e^{-\frac{m\omega}{2\hbar} x^2}$$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$ Hermite Polynomial

$H_0(x) = 1$

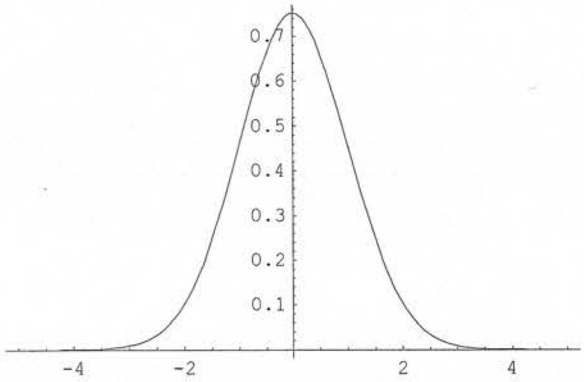
$H_1(x) = 2x$

$H_2(x) = 4x^2 - 2$

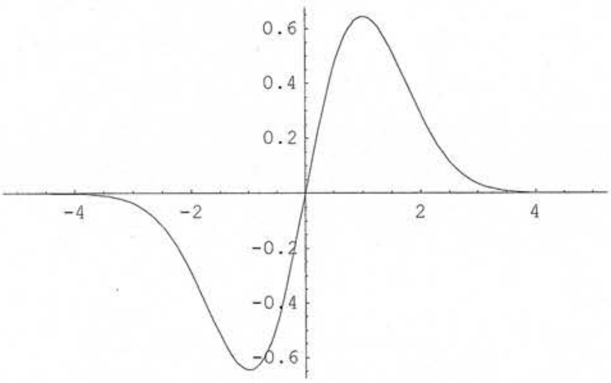
$H_3(x) = 8x^3 - 12x$

p240 예제 5.7

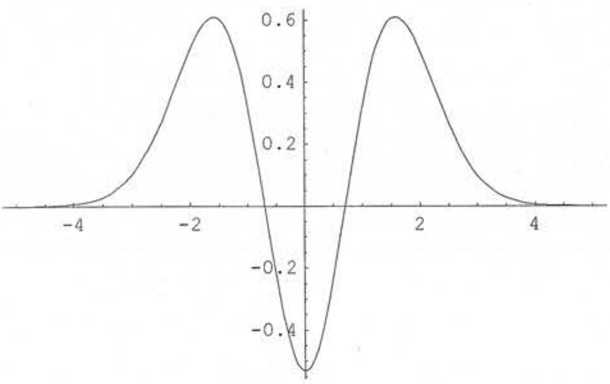
```
In[1]:= m = 1; w = 1; hbar = 1;

In[2]:= phi[n_, x_] := 1 / Sqrt[2^n Gamma[n + 1]] ((m w) / (Pi hbar)) ^ (1 / 4)
          HermiteH[n, Sqrt[m w / hbar] x] Exp[- ((m w) / (2 hbar)) x^2]

In[3]:= amp[n_] := (2 / (m w^2)) (n + 1 / 2) hbar w

In[4]:= cla[n_, x_] := 1 / (amp[n]^2 w^2 - w^2 x^2)

In[13]:= Plot[phi[0, x], {x, -5, 5}]
```




```
Out[13]= - Graphics -

In[14]:= Plot[phi[1, x], {x, -5, 5}, PlotRange -> All]
```

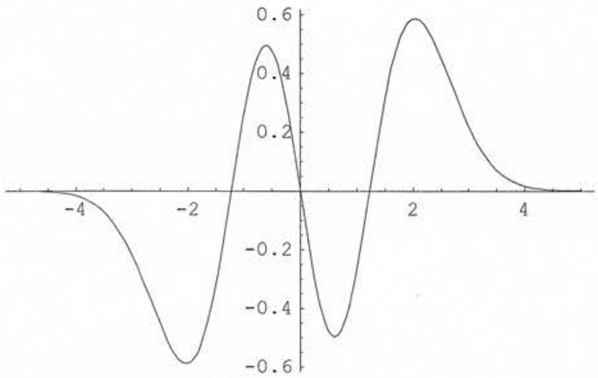



```
Out[14]= - Graphics -

In[15]:= Plot[phi[2, x], {x, -5, 5}, PlotRange -> All]
```



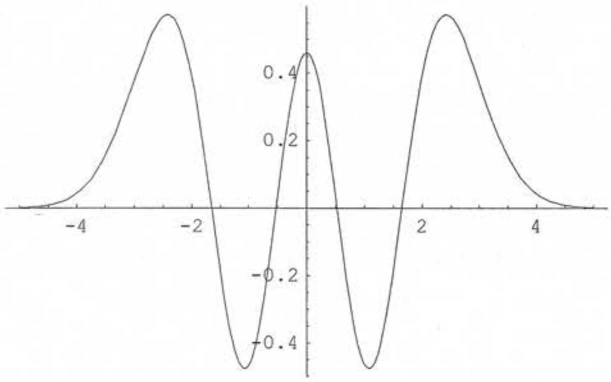

```
Out[15]= - Graphics -
```

In[16]:= **Plot[phi[3, x], {x, -5, 5}, PlotRange -> All]**



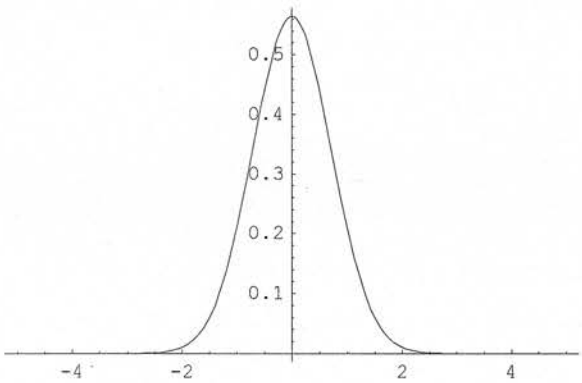
Out[16]= - Graphics -

In[17]:= **Plot[phi[4, x], {x, -5, 5}, PlotRange -> All]**



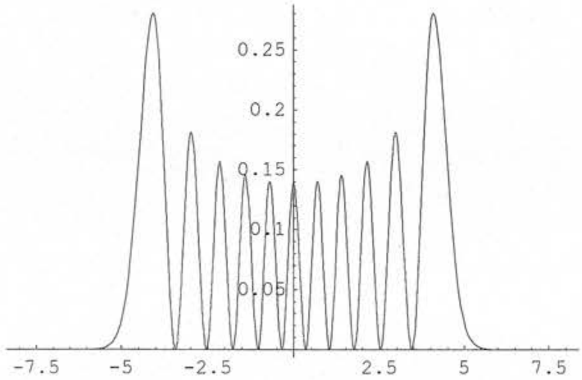
Out[17]= - Graphics -

In[19]:= **Plot[phi[0, x]^2, {x, -5, 5}, PlotRange -> All]**



Out[19]= - Graphics -

In[21]:= **Plot[phi[10, x]^2, {x, -8, 8}, PlotRange -> All]**



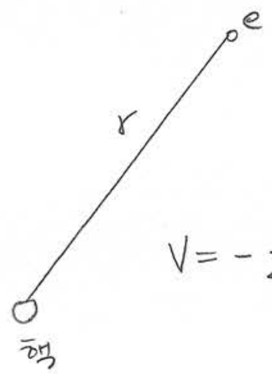
Out[21]= - Graphics -

CH6. 수소원자의 양자론

§. 수소원자의 Schrödinger 방정식

Schrödinger Eq.

$$\left[-\frac{\hbar^2}{2m}\nabla^2+V\right]\Psi = E\Psi$$



$$V=-\frac{1}{4\pi\epsilon_0}\frac{e^2}{r}$$

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$: retangular coordinate

$$=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}$$: cylindrical coordinate

$$=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

: spherical coordinate

We choose a spherical coordinate.

$$\left[-\frac{\hbar^2}{2m}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2}\right\}\right.$$

$$\left.-\frac{1}{4\pi\epsilon_0}\frac{e^2}{r}\Psi\right]=E\Psi.$$

§ 변수 분리

Let $\psi = R(r)\,\Theta(\theta)\,\Phi(\phi)$

Then Schrödinger equation becomes

$$\Theta\Phi \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + R\Phi\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + R\Theta\frac{1}{r^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2}$$

$$+\frac{2m}{\hbar^2}\left(E+\frac{1}{4\pi\epsilon_0}\frac{e^2}{r}\right)R\Theta\Phi = 0 \qquad \| \times \frac{1}{R\Theta\Phi}$$

$$\Rightarrow \frac{1}{Rr^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)$$

$$+\frac{1}{\Phi r^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2} + \frac{2m}{\hbar^2}\left(E+\frac{1}{4\pi\epsilon_0}\frac{e^2}{r}\right) = 0 \qquad \| \times r^2\sin^2\theta$$

$$\Rightarrow \underbrace{\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{2m r^2\sin^2\theta}{\hbar^2}\left(E+\frac{1}{4\pi\epsilon_0}\frac{e^2}{r}\right)}_{r\text{과 }\theta\text{의 함수}} = \underbrace{-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}}_{\phi\text{의 함수}} \equiv m^2$$

$$\Rightarrow \quad \frac{d^2\Phi}{d\phi^2} = -m^2 \Phi$$

$$\Rightarrow \quad \frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)$$

$$+ \frac{2mr^2\sin^2\theta}{\hbar^2}\left(E + \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}\right) = m^2 \quad \Big| \times \frac{1}{\sin^2\theta}$$

r의 함수

$$\underline{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{2mr^2}{\hbar^2}\left(E + \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}\right)}$$

$$= \underline{\frac{m^2}{\sin^2\theta} - \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)} \equiv l(l+1)$$

θ의 함수

$$\Rightarrow \quad \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left[\frac{2m}{\hbar^2}\left(E + \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}\right) - \frac{l(l+1)}{r^2}\right]R = 0$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0$$

$$\frac{d^2\Phi}{d\phi^2} = -m^2\Phi$$

§ solution

(i) $\frac{d^2\Phi}{d\phi^2} = -m^2\Phi$

$$\Rightarrow \Phi = A e^{im\phi}$$

$$\Phi(\phi=0) = \Phi(\phi=2\pi)$$

$m = 0, \pm 1, \pm 2, \cdots$:

(ii) $\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0$

$$\Rightarrow \frac{d^2\Theta}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{d\Theta}{d\theta} + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0$$

$$x = \cos\theta$$

$$\frac{d}{d\theta} = -\sqrt{1-x^2}\,\frac{d}{dx}$$

$$\frac{d^2}{d\theta^2} = (1-x^2)\frac{d^2}{dx^2} - x\frac{d}{dx}$$

$$\Rightarrow (1-x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + \left[l(l+1) - \frac{m^2}{1-x^2}\right]\Theta = 0$$

associated Legendre 미분방정식

$$\Theta = P_l^m(\cos\theta) = (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_l(x) \Big|_{x=\cos\theta}$$

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l}(x^2-1)^l$$: Legendre polynomial

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x)$$

$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$

$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$, $P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$

$m \leq l$

Thus we have

$$\boxed{\begin{array}{l} l = 0, 1, 2, \cdots \\ m = 0, \pm 1, \pm 2, \cdots, \pm m \end{array}}$$

(iii) $$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\left[\frac{2m}{\hbar^2}\left(E+\frac{1}{4\pi\epsilon_0}\frac{e^2}{r}\right)-\frac{l(l+1)}{r^2}\right]R=0$$

$$\Rightarrow \frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\left[\frac{2m}{\hbar^2}\left(E+\frac{1}{4\pi\epsilon_0}\frac{e^2}{r}\right)-\frac{l(l+1)}{r^2}\right]R=0$$

Note that $E<0$ due to bound state.

$$\rho \equiv \sqrt{\frac{8m|E|}{\hbar^2}}\, r$$

$$\Rightarrow \frac{d^2R}{d\rho^2}+\frac{2}{\rho}\frac{dR}{d\rho}-\frac{l(l+1)}{\rho^2}+\left(\frac{\lambda}{\rho}-\frac{1}{4}\right)R=0 \qquad (1)$$

$$\lambda=\frac{e^2}{4\pi\epsilon_0\hbar}\sqrt{\frac{m}{2|E|}}$$

Introduce fine structure constant

$$\boxed{\alpha=\frac{e^2}{4\pi\epsilon_0 c\hbar}=\frac{1}{137}}$$

$$\lambda=\alpha\sqrt{\frac{mc^2}{2|E|}} \qquad (2)$$

At $\rho\sim\infty$ $\quad R\sim e^{-\frac{1}{2}\rho}$

Put
$$R=e^{-\frac{1}{2}\rho}\,G(\rho) \qquad (3)$$

(3) → (1)

$$\frac{d^2G}{d\rho^2} - \left(1 - \frac{2}{\rho}\right)\frac{dG}{d\rho} + \left[\frac{\lambda-1}{\rho} - \frac{l(l+1)}{\rho^2}\right] G = 0 \qquad (4)$$

$\rho = 0$: regular singular point

$$G = \rho^{\alpha} \sum_{n=0}^{\infty} C_n \rho^n \qquad (5)$$

(5) → (4)

$$\alpha = l$$

$$C_1 = \frac{(l+1) - \lambda}{(2l+2)} C_0$$

$$C_2 = \frac{[(l+1)-\lambda][(l+2)-\lambda]}{2!\,(2l+2)\,(2l+3)} C_0$$

$$C_3 = \frac{[(l+1)-\lambda][(l+2)-\lambda][(l+3)-\lambda]}{3!\,(2l+2)\,(2l+3)\,(2l+4)} C_0$$

$$\vdots$$

∴

$$R(\rho) = A e^{-\frac{1}{2}\rho} \rho^{l} \left[1 - \frac{\lambda - (l+1)}{2l+2}\rho + \frac{[\lambda-(l+1)][\lambda-(l+2)]}{2!\,(2l+2)\,(2l+3)}\rho^2 - \frac{[\lambda-(l+1)][\lambda-(l+2)][\lambda-(l+3)]}{3!\,(2l+2)\,(2l+3)\,(2l+4)}\rho^3 + \cdots \right] \qquad (6)$$

⇒ 확률 해석 위해서 $R(\rho=\infty) = 0$

⇒ Series must be terminated !!

$$\lambda = l + 1 + n_r \qquad n_r = 0, 1, 2, \cdots$$

$$\equiv n \qquad n = 1, 2, 3, \cdots$$

$$E_n = -\frac{mc^2\alpha^2}{2n^2} = -\frac{me^4}{8\varepsilon_0^2 h^2}\frac{1}{n^2} = -13.6\,eV\,\frac{1}{n^2}$$

Summary of quantum number

$n = 1, 2, 3, \cdots$ principal quantum number

$l = 0, 1, \cdots, (n-1)$ orbital quantum number

$m = 0, \pm 1, \pm 2, \cdots \pm l$ magnetic quantum number

$$R_{n,l}(\rho) = A e^{-\frac{1}{2}\rho} \rho^l \left[1 - \frac{n-(l+1)}{2l+2}\rho + \frac{[n-(l+1)][n-(l+2)]}{2!\,(2l+2)(2l+3)}\rho^2 + \cdots \right]$$

$$\int_0^\infty dr\, r^2 |R_{n,l}(r)|^2 = 1$$

(Ex) $R_{10} = \dfrac{2}{a_0^{\frac{3}{2}}} e^{-\frac{r}{a_0}}$

$a_0 = \dfrac{4\pi\varepsilon_0 \hbar^2}{me^2} = 5.3 \times 10^{-11}\,m$

Bohr radius

$$R_{20} = \frac{1}{2\sqrt{2}\,a_0^{\frac{3}{2}}}\left(2 - \frac{r}{a_0}\right) e^{-\frac{r}{2a_0}}$$

p.238 풀이

$$R_{21} = \frac{1}{2\sqrt{6}\,a_0^{\frac{3}{2}}}\frac{r}{a_0} e^{-\frac{r}{2a_0}}$$

p251

$\vdots$

$$Y_{l,m}(\theta,\phi) = \Theta(\theta)\,\Phi(\phi)$$

$$= (-1)^m \left[\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!} \right]^{\frac{1}{2}} P_l^m(\cos\theta)\, e^{im\phi}$$

$$Y_{l,-m}(\theta,\phi) = (-1)^m\, Y^*_{l,m}(\theta,\phi)$$

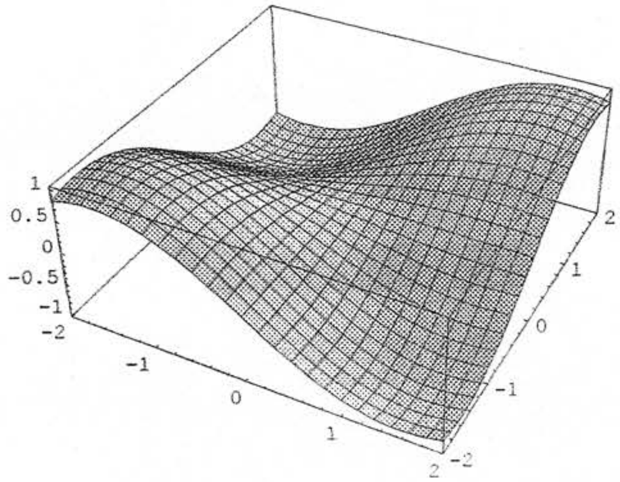
Spherical Harmonics

$$Y_{0,0} = \sqrt{\frac{1}{4\pi}}, \quad Y_{1,1} = -\sqrt{\frac{3}{8\pi}}\, e^{i\phi}\sin\theta, \quad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{1,-1} = \sqrt{\frac{3}{8\pi}}\, e^{-i\phi}\sin\theta, \quad Y_{2,2} = \sqrt{\frac{15}{32\pi}}\, e^{2i\phi}\sin^2\theta$$

$$Y_{2,1} = -\sqrt{\frac{15}{8\pi}}\, e^{i\phi}\sin\theta\cos\theta, \quad Y_{2,0} = \sqrt{\frac{5}{16\pi}}\,(3\cos^2\theta - 1) \quad \cdots$$

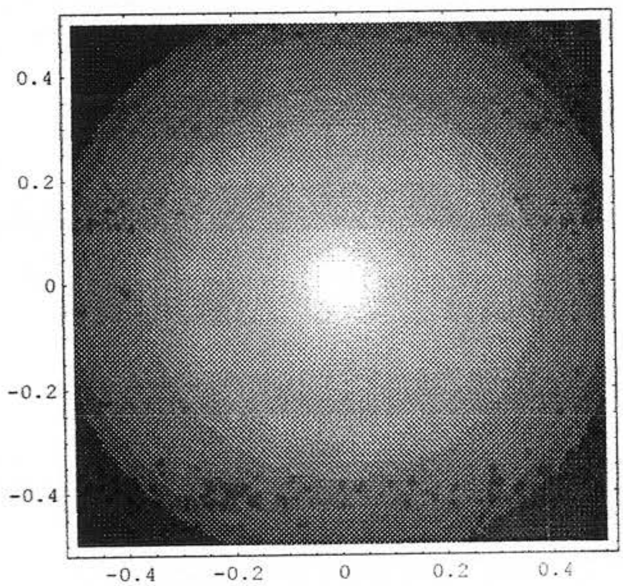
In[13]:=
```
Plot3D[Sin[x] Sin[y], {x, -2, 2}, {y, -2, 2}]
```



Out[13]= - SurfaceGraphics -

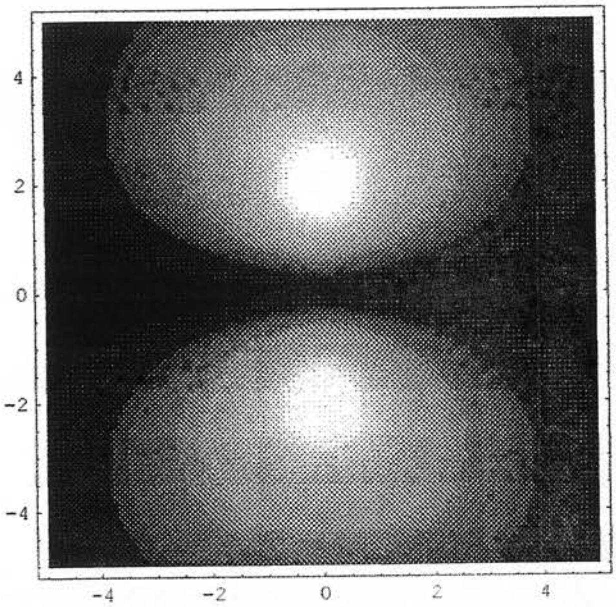
In[31]:=
```
psi1[x_, y_, z_] := Exp[-Sqrt[x^2 + y^2 + z^2]] / Sqrt[Pi];
psi200[x_, y_, z_] :=
  (2 - Sqrt[x^2 + y^2 + z^2]) Exp[-Sqrt[x^2 + y^2 + z^2] / 2] / (4 Sqrt[2 Pi]);
psi210[x_, y_, z_] := Sqrt[x^2 + y^2 + z^2] Exp[-Sqrt[x^2 + y^2 + z^2] / 2]
   z / (4 Sqrt[2 Pi] Sqrt[x^2 + y^2 + z^2]);
psi211[x_, y_, z_] := Sqrt[x^2 + y^2 + z^2] Exp[-Sqrt[x^2 + y^2 + z^2] / 2]
  Sqrt[x^2 + y^2] (x + I y) / (8 Sqrt[Pi] Sqrt[x^2 + y^2 + z^2] Sqrt[x^2 + y^2])
```

In[37]:=
```
DensityPlot[Abs[psi1[0, y, z]]^2, {y, -0.5, 0.5},
  {z, -0.5, 0.5}, Mesh → False, PlotPoints → 100]
```



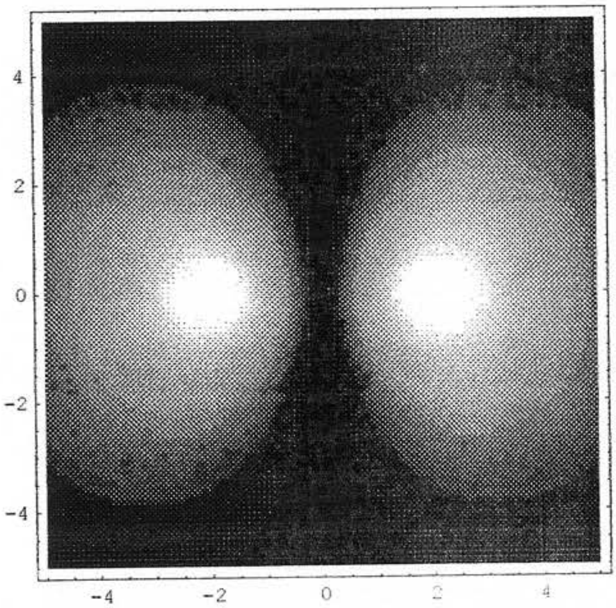
Out[37]= - DensityGraphics -

```
In[38]:= DensityPlot[Abs[psi210[0, y, z]]^2,
          {y, -5, 5}, {z, -5, 5}, Mesh → False, PlotPoints → 100]
```



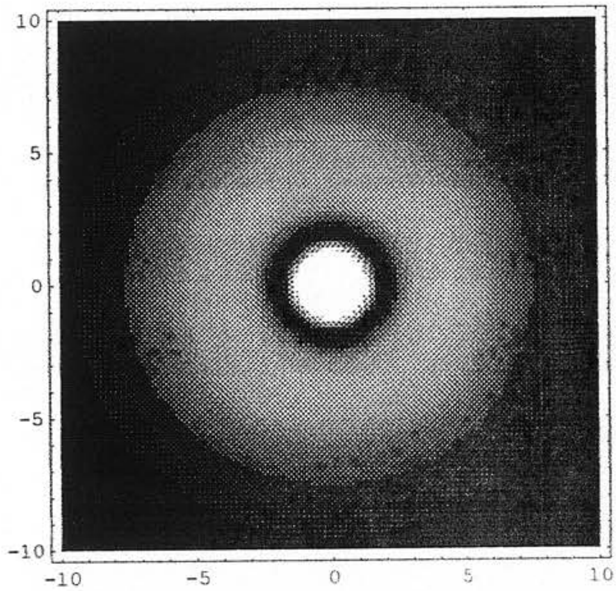

```
Out[38]= - DensityGraphics -
```

```
In[39]:= DensityPlot[Abs[psi211[0, y, z]]^2,
          {y, -5, 5}, {z, -5, 5}, Mesh → False, PlotPoints → 100]
```




```
Out[39]= - DensityGraphics -
```

In[42]:=
```
DensityPlot[Abs[psi200[0, y, z]]^2, {y, -10, 10},
  {z, -10, 10}, Mesh → False, PlotPoints → 100]
```



Out[42]= - DensityGraphics -

CH9. 통계역학

§ 분포 함수

$n(\epsilon)$: 에너지 ϵ 을 가지는 입자수

$g(\epsilon)$: 에너지가 ϵ 인 상태수 (statistical weight)

$f(\epsilon)$: 분포 함수 (distribution function)

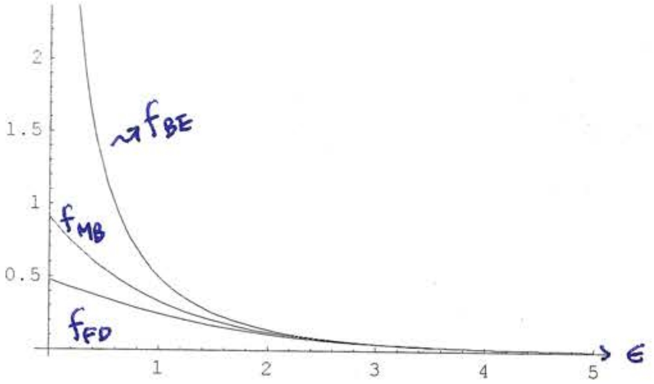
각각 에너지 ϵ 인 상태에 있는 평균 입자수

$$\boxed{n(\epsilon) = g(\epsilon) f(\epsilon)}$$

$f(\epsilon) \propto$ 에너지 ϵ 인 상태가 점유될 확률

	Maxwell-Boltzmann	Bose-Einstein	Fermi-Dirac
$f(\epsilon)$	$Ae^{-\frac{\epsilon}{kT}}$	$\frac{1}{e^{\alpha}e^{\frac{\epsilon}{kT}}-1}$	$\frac{1}{e^{\alpha}e^{\frac{\epsilon}{kT}}+1}$
적용되는 계	동일, 구별가능 입자	동일, 구별불가능 입자이며 Pauli의 배타원리를 따르지 않는다.	동일, 구별 불가능 입자이며 Pauli의 배타원리를 따른다
입자의 범주	Classical	Boson	fermion
입자의 성질	임의의 스핀, 입자들이 충분히 떨어져 있으므로 파동방정식의 중첩이 없다	spin = 0, 1, 2, ⋯ wave function is symmetric under the interchange of particles	spin = $\frac{1}{2}$, $\frac{3}{2}$, ⋯ wave function is anti-symmetric under the interchange of particles
예	이상기체	액체 Helium photon, phonon	free electron in metal 백색 왜성 내 전자

```
In[9]:= Plot[{Exp[-0.1] Exp[-x], 1 / (Exp[0.1] Exp[x] - 1), 1 / (Exp[0.1] Exp[x] + 1)}, {x,
        0, 5}, PlotStyle → {RGBColor[0, 1, 0], RGBColor[0, 0, 1], RGBColor[1, 0, 0]}]
```



$kT = 1$

$\alpha = 0.1$

$A = e^{-0.1}$

```
Out[9]= - Graphics -
```

k: Boltzmann constant

$k = 1.381 \times 10^{-23}$ Joule/K $= 8.617 \times 10^{-5}$ eV/K

§ Maxwell-Boltzmann Statistics

p354 문제

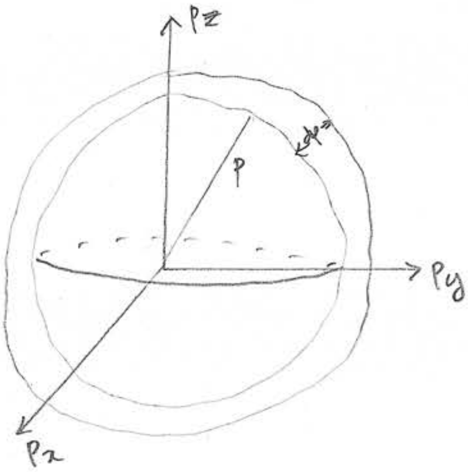
§ ideal Gas : Application of Maxwell - Boltzmann statistics

$$\boxed{\begin{array}{l} \text{ideal Gas} \\ \text{평균 운동에너지} = \frac{3}{2}kT \\ \text{상태방정식: } PV = NkT \end{array}}$$

$$n(\epsilon)\,d\epsilon = (g(\epsilon)\,d\epsilon)\,f(\epsilon) = (g(\epsilon)\,d\epsilon)\,A e^{-\frac{\epsilon}{kT}} \qquad (1)$$

$$\epsilon = \frac{\vec{p}^{\,2}}{2m} = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2)$$

$$\epsilon \sim \epsilon + d\epsilon \quad \Leftrightarrow \quad p \sim p + dp$$



$$g(p)\,dp \propto 4\pi p^2\,dp$$

Thus let

$$g(p)\,dp = B p^2\,dp$$

$$\therefore \quad g(\epsilon)\,d\epsilon = B p^2\,dp \qquad (2)$$

Since $p^2 = 2m\epsilon$

$$2p\,dp = 2m\,d\epsilon$$

$$dp = \frac{m\,d\epsilon}{\sqrt{2m\epsilon}} \qquad (3)$$

(3) → (2)

$$g(\epsilon)d\epsilon = B\sqrt{\epsilon}\, d\epsilon \qquad (4)$$

(4) → (1)

$$n(\epsilon)d\epsilon = C\sqrt{\epsilon}\, e^{-\frac{\epsilon}{kT}}\, d\epsilon \qquad (5)$$

Let total number of particles be N. Thus we get

$$N \equiv \int_0^\infty n(\epsilon)\, d\epsilon \qquad (6)$$

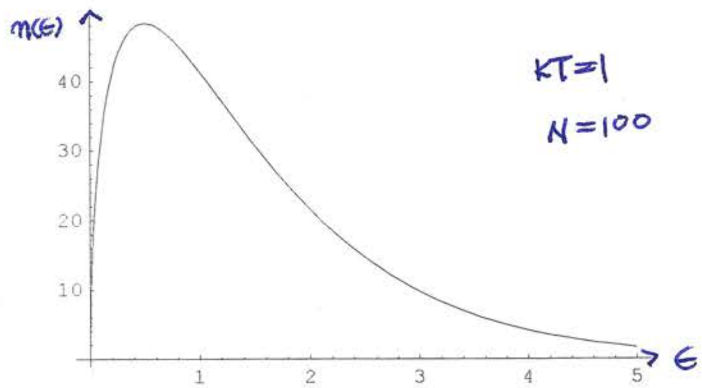
Then we can determine C

$$C = \frac{2\pi N}{(\pi kT)^{\frac{3}{2}}} \qquad (7)$$

(7) → (5)

$$n(\epsilon)d\epsilon = \frac{2\pi N}{(\pi kT)^{\frac{3}{2}}} \sqrt{\epsilon}\, e^{-\frac{\epsilon}{kT}}\, d\epsilon \qquad (8)$$

In[14]:= Plot[2 Pi 100 / (Pi)^(3/2) Sqrt[x] Exp[-x], {x, 0, 5}]



Out[14]= - Graphics -

내부에너지 : E

$$E = \int_0^{\infty} \epsilon\, n(\epsilon)\, d\epsilon = \frac{3}{2} N k T \quad (9)$$

평균 운동에너지 : $\bar{\epsilon}$

$$\bar{\epsilon} = \frac{E}{N} = \frac{3}{2} k T \quad (10)$$

$$\boxed{\bar{\epsilon} = \frac{1}{2} k T \times \text{degree of freedom}}$$

속력 분포

$$\epsilon = \frac{1}{2} m v^2$$

$$d\epsilon = m v\, dv$$

$$\therefore n(v)\, dv = \frac{\sqrt{2}\, \pi N m^{\frac{3}{2}}}{(\pi k T)^{\frac{3}{2}}} v^2 e^{-\frac{m v^2}{2kT}} dv \quad (11)$$

(i) average speed : $\bar{v}$

$$\bar{v} \equiv \frac{1}{N} \int_0^{\infty} v\, n(v)\, dv = \sqrt{\frac{8kT}{m\pi}} \quad (12)$$

(ii) root-mean-square speed : v_{rms}

$$\frac{1}{2} m v_{rms}^2 \equiv \frac{3}{2} k T$$

$$\Rightarrow v_{rms} = \sqrt{\frac{3kT}{m}} \quad (13)$$

(iii) most probable speed : v_p

$$\left. \frac{\partial n(v)}{\partial v} \right|_{v = v_p} \equiv 0$$

$$\Rightarrow v_p = \sqrt{\frac{2kT}{m}} \quad (14).$$

Note that $v_p < \bar{v} < v_{rms}$ in ideal gas.

§ Wave function for boson and fermion

state	particle		
a	1	2	$\psi = \psi_a(1)\,\psi_b(2)$
b	2	1	$\psi = \psi_a(2)\,\psi_b(1)$

If particle 1 and 2 are indistinguishable,

$$\psi_B = \frac{1}{\sqrt{2}}\left[\psi_a(1)\,\psi_b(2) + \psi_a(2)\,\psi_b(1)\right] \qquad (1)$$

$1 \leftrightarrow 2$ symmetric

for boson

$$\psi_F = \frac{1}{\sqrt{2}}\left[\psi_a(1)\,\psi_b(2) - \psi_a(2)\,\psi_b(1)\right] \qquad (2)$$

$1 \leftrightarrow 2$ anti-symmetric

for fermion

If $a=b$, $\psi_B = \sqrt{2}\,\psi_a(1)\,\psi_a(2)$

$\psi_F = 0$: Pauli exclusion principle.

Distribution function for fermion

$$f_{FD}(\epsilon) = \frac{1}{e^{\alpha} e^{\frac{\epsilon}{kT}} + 1} \qquad (3)$$

Let

$$\underline{\epsilon_F \equiv -\alpha kT} \qquad (4)$$

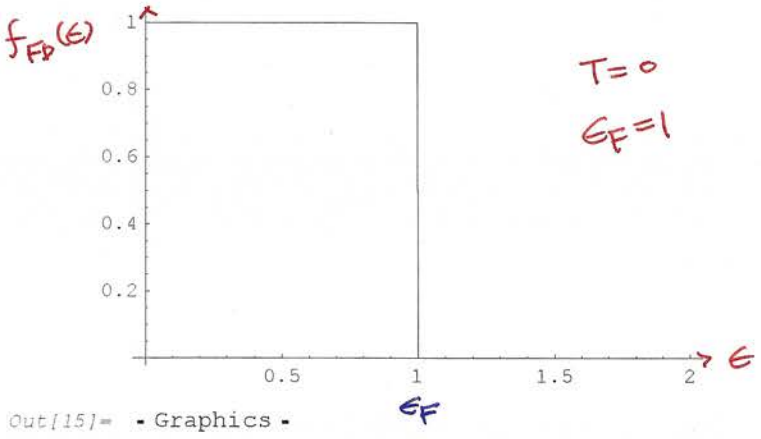
fermi energy

note) $f_{FD}(\epsilon = \epsilon_F) = \frac{1}{2}$

From (3) and (4)

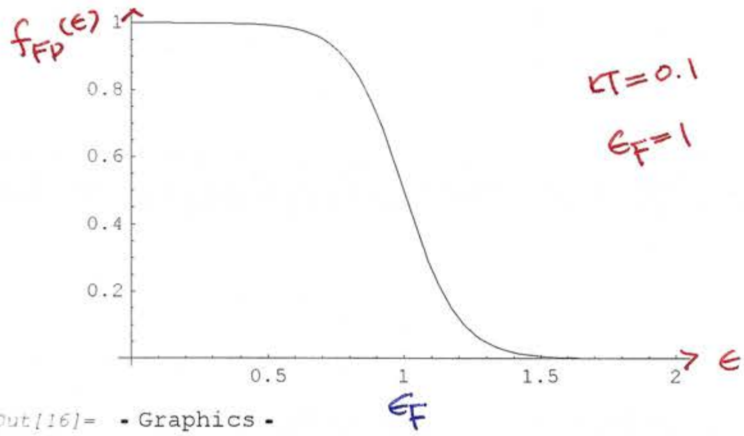
$$f_{FD}(\epsilon) = \frac{1}{e^{\frac{\epsilon - \epsilon_F}{kT}} + 1} \qquad (5)$$

In[15]:= `Plot[1 / (Exp[(x - 1) / 0.00001] + 1), {x, 0, 2}]`



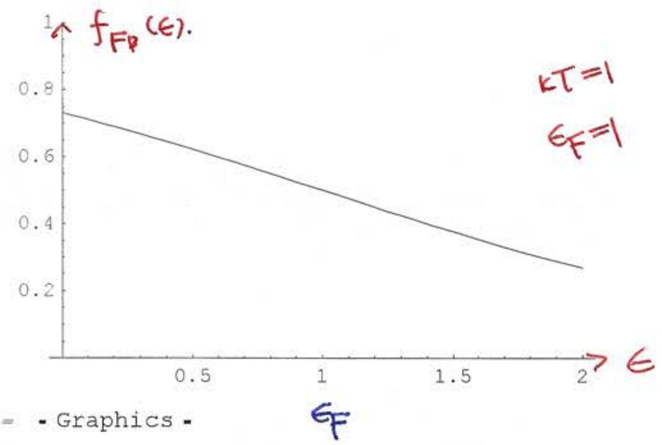
Out[15]= - Graphics -

In[16]:= `Plot[1 / (Exp[(x - 1) / 0.1] + 1), {x, 0, 2}]`



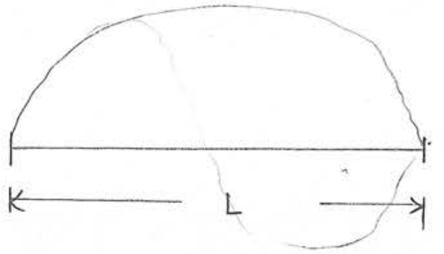
Out[16]= - Graphics -

In[18]:= `Plot[1 / (Exp[(x - 1) / 1] + 1), {x, 0, 2}, PlotRange → {0, 1}]`



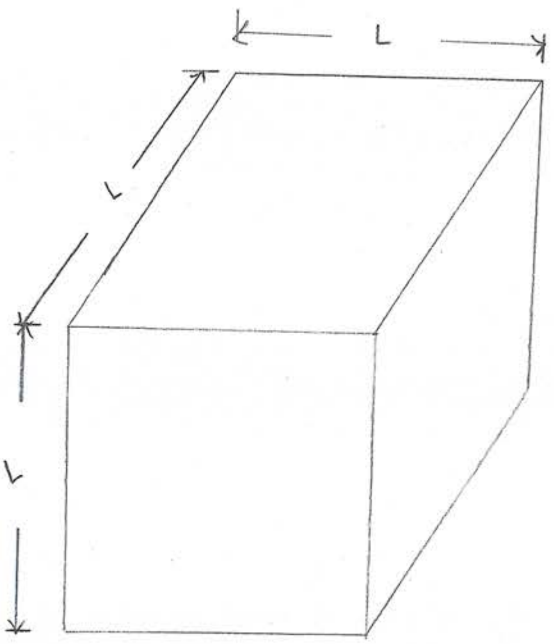
Out[18]= - Graphics -

§ black-body radiation



$n = \frac{2L}{\lambda} \quad n = 1, 2, \cdots$

정상파 조건



$j_x^2 + j_y^2 + j_z^2 = \left(\frac{2L}{\lambda}\right)^2$

$j_x = 0, 1, 2, \cdots$ 정상파 조건

$j_y = 0, 1, 2, \cdots$

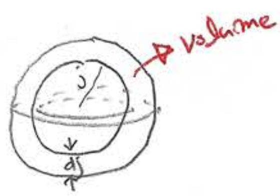
$j_z = 0, 1, 2, \cdots$

* Rayleigh - Jeans 공식

Rayleigh - Jeans : black - body radiation is standing - wave.

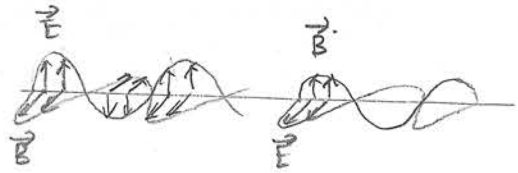
$$g(j)\,dj = 2 \times \frac{1}{8} \times 4\pi j^2\,dj = \pi j^2\,dj \qquad (1)$$

$4\pi j^2\,dj \Rightarrow$



$\frac{1}{8} \Rightarrow j_x, j_y, j_z > 0$

$2 \Rightarrow$ Polarization



$$\left.\begin{aligned} j &= \frac{2L}{\lambda} = \frac{2L\nu}{c} \\ dj &= \frac{2L}{c}\, d\nu \end{aligned}\right\} \quad (2)$$

(2) ⇒ (1)

$$g(\nu)\, d\nu = \frac{8\pi L^3}{c^3}\, \nu^2\, d\nu \qquad (3)$$

$G(\nu)\,d\nu$: density of standing wave in black body

$$G(\nu)\, d\nu = \frac{1}{L^3}\, g(\nu)\, d\nu = \frac{8\pi\nu^2}{c^3}\, d\nu \qquad (4)$$

$u(\nu)\,d\nu$: 단위 부피당 에너지

$$\boxed{u(\nu)\, d\nu = \bar{\epsilon}\, G(\nu)\, d\nu = \underline{kT}\, G(\nu)\, d\nu = \frac{8\pi\nu^2 kT}{c^3}\, d\nu} \qquad (5)$$

$\frac{1}{2}kT \times$ degree of freedom

$= \frac{1}{2}kT \times 2$

Kinetic + potential

Rayleigh - Jeans formula

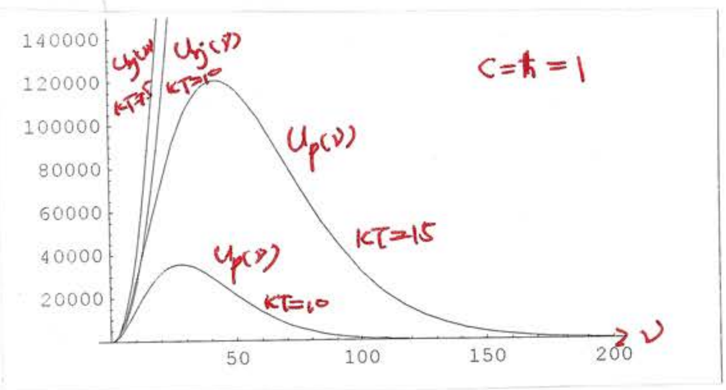
Planck

$$f_{BE} = \frac{1}{e^{\frac{h\nu}{kT}} - 1} \qquad (\alpha = 0)$$

$$\Rightarrow u(\nu)\,d\nu = h\nu\, G(\nu)\, f_{BE}\, d\nu$$

$$= \frac{8\pi h}{c^3} \frac{\nu^3}{e^{\frac{h\nu}{kT}} - 1}\, d\nu \qquad (6)$$

Planck formula



From (6)

$$\frac{du(\nu)}{d\nu} = 0$$

$$\frac{hc}{kT\lambda_{max}} = 4.965 \quad (? \; 2.821)$$

$$\Rightarrow \lambda_{max} T = \frac{hc}{4.965\,k} = 2.9 \times 10^{-3} \; m \cdot K$$

Wien 법칙

Integral formula

$$\boxed{\int_0^{\infty} \frac{x^{\nu-1}}{e^{\mu x}-1}\,dx = \frac{1}{\mu^{\nu}}\Gamma(\nu)\,\zeta(\nu)}$$

Using $\zeta(4) = \frac{\pi^4}{90}$,

$$u = \int_0^{\infty} u(\nu)\,d\nu = aT^4$$: 단위 volume 당 total energy.

$$a = \frac{8\pi^5 k^4}{15c^3h^3}$$: universal constant

R: radiated energy by a black body per second per unit area

$$R = e\sigma T^4$$

$$\sigma = \frac{ac}{4} = 5.670\times10^{-8}\ \mathrm{W/m^2\,K^4}$$

$$e = \begin{cases} 0 & \text{완전반사체} \\ 1 & \text{흑체} \end{cases}$$

Stefan-Boltzmann 법칙

⇒ P377 문제